Math 8174: Homework 2

Due February 4, 2009

1. Let C_n be the cyclic group of order n.
   (a) Completely classify the irreducible C_n-modules,
   (b) Show how to decompose the regular module $\mathbb{C}C_n$ in terms of irreducibles. That is, find an explicit basis $\mathcal{B} = \{b_1, \ldots, b_n\} \subseteq \mathbb{C}C_n$ such that for each $1 \leq j \leq n$, $\mathbb{C}$-span$\{b_j\}$ is a submodule of $\mathbb{C}C_n$.

2. Let V be a completely reducible A-module. Prove the converse of Schur's Lemma. That is, if for every A-module homomorphism $\theta : V \to V$ there exists $c \in \mathbb{C}$ such that $\theta(v) = cv$, then V is irreducible.

3. (a) Find an infinite group G and a G-module V such that V is not completely reducible (ie. show that Maschke's Theorem does not apply to infinite groups),
   (b) Find a finite dimensional $\mathbb{C}$-algebra A whose regular module is not completely reducible (ie. find a non-semisimple algebra).

   Hints are certainly available on request.

4. Let A be a semisimple algebra. Suppose $V \subseteq A$ is a submodule of the regular module A.
   (a) Show that there exists an idempotent $e \in A$ (an element such that $e^2 = e$) such that
   $$V = Ae.$$
   (b) Show that if $\theta \in \mathrm{Hom}_A(V, A)$, then
   $$\theta(v) = va, \qquad \text{for some } a \in A.$$
   (c) Show that
   $$\mathrm{End}_A(V) \cong eAe,$$
   as vector spaces.

   Remark: One can show that the two spaces in (c) are in fact anti-isomorphic as algebras.