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Math 8174: Homework 1

Due: September 8, 2010

1. Find the tangent space to $\mathrm{SL}_n(\mathbb{R})$ as an explicit subspace of $\mathfrak{gl}_n(\mathbb{R})$.

2. Find the tangent space to either $O_n(\mathbb{R})$ or $\mathrm{Sp}_{2n}(\mathbb{R})$ as an explicit subspace of $\mathfrak{gl}_n(\mathbb{R})$.

3. This problem classifies all dimension 3 Lie algebras $\mathfrak{g}$ with $\dim([\mathfrak{g},\mathfrak{g}]) = 2$.

   (a) Show that $[\mathfrak{g},\mathfrak{g}]$ is commutative.

   (b) If $x \in \mathfrak{g}$ is such that $\mathfrak{g} = \mathbb{C}\text{-span}\{x\} + [\mathfrak{g},\mathfrak{g}]$, then show that $\mathrm{ad}_x : [\mathfrak{g},\mathfrak{g}] \to [\mathfrak{g},\mathfrak{g}]$ is an isomorphism.

   (c) Classify all the dimension 3 Lie algebras $\mathfrak{g}$ with $\dim([\mathfrak{g},\mathfrak{g}]) = 2$.

   Hint: For (c), split into cases depending on whether ad_x is diagonalizable or not.

4. Show that Lie's Theorem fails for fields of characteristic $p > 0$. Hint: An example in two dimensions works.

5. The Virasoro Lie algebra $\mathfrak{v}$ is given by

   $$\mathfrak{v} = \mathbb{C}\text{-span}\{v_n \mid n \in \mathbb{Z}\},$$

   with

   $$[v_m, v_n] = (n-m)v_{m+n}.$$

   (a) Show that $\mathfrak{v}$ is generated as a Lie algebra by two elements,

   (b) Show that $\mathfrak{v}$ has no nonzero, proper ideal (ie. it is simple).