Math 6350: Homework 5

Due: Friday, November 9

A. The following exercises make use of (and define) a sequence \( \{B_n\}_{n \geq 1} \) known as the Bernoulli numbers.

V.1.3.4. Show that close to zero,

\[
\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n \geq 1} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n-1},
\]

for some (unspecified) integers \( B_1, B_2, \ldots \in \mathbb{Z} \). Calculate \( B_1, B_2, \) and \( B_3 \).

V.2.1.5. Show that

\[
\sum_{n \geq 1} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}.
\]

B.

V.2.1.4. Find a closed form for

\[
\sum_{n=1}^{\infty} \frac{1}{(z + n)^2 + a^2}.
\]

V.2.2.1. Show that

\[
\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.
\]

C. (1) Prove uniqueness in Weierstrass’ Theorem.

V.2.3.2. Prove that

\[
\sin(\pi(z + k)) = e^{\pi z \cot(\pi k)} \prod_{n=-\infty}^{\infty} \left(1 + \frac{z}{n + k}\right) e^{-\frac{z}{n+k}}, \quad \text{for } k \in \mathbb{Z}.
\]

D.

V.2.4.3. What are the residues of \( \Gamma(z) \).

(2) Show that for \( x > 0 \),

\[
e^{-x} = \frac{1}{2\pi i} \int_{\gamma} x^{-s} \Gamma(s) ds,
\]

where \( \gamma(t) = 1 + it, -\infty \leq t \leq \infty \).