Math 6350: Homework 5

Due: Friday, November 9

A. The following exercises make use of (and define) a sequence $\{B_n\}_{n\geq 1}$ known as the *Bernoulli numbers.*

V.1.3.4. Show that close to zero,

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n\geq 1} (-1)^{n-1} \frac{B_n}{(2n)!} z^{2n-1},$$

for some (unspecified) integers $B_1, B_2, \ldots \in \mathbb{Z}$. Calculate B_1, B_2, and B_3.

V.2.1.5. Show that

$$\sum_{n\geq 1} \frac{1}{n^{2k}} = 2^{2k-1} \frac{B_k}{(2k)!} \pi^{2k}.$$

B.

V.2.1.4. Find a closed form for

$$\sum_{-\infty}^{\infty} \frac{1}{(z+n)^2 + a^2}.$$

V.2.2.1. Show that

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \frac{1}{2}.$$

C. (1) Prove uniqueness in Weierstrass' Theorem.

V.2.3.2. Prove that

$$\sin(\pi(z+k)) = e^{\pi z \cot(\pi k)} \prod_{-\infty}^{\infty} \left(1 + \frac{z}{n+k}\right) e^{-\frac{z}{n+k}}, \qquad \text{for } k \in \mathbb{Z}.$$

D.

V.2.4.3. What are the residues of $\Gamma(z)$.

(2) Show that for $x > 0$,

$$e^{-x} = \frac{1}{2\pi i} \int_{\gamma} x^{-s} \Gamma(s) ds,$$

where $\gamma(t) = 1 + it$, $-\infty \leq t \leq \infty$.