Math 6350: Homework 3

Due: Friday, October 5

A. Let

\[ \Delta_1 = \{ z \in \mathbb{C} \mid |z| < 2 \} \]
\[ \Delta_2 = \left\{ z \in \mathbb{C} \mid |z - i| < \frac{1}{2} \right\}. \]

Evaluate the following integrals.

(1) \[ \int_{\partial \Delta_1} \frac{dz}{z^4 - 1}, \]
(2) \[ \int_{\partial \Delta_2} \frac{dz}{z^4 - 1}, \]
(3) \[ \int_{\partial \Delta_1} z^m (1 - z)^n dz, \]
(4) \[ \int_{\partial \Delta_2} \log(|z|). \]

B. Suppose that \( f \) is holomorphic in the open unit disk \( \Delta = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

(1) Suppose that there is a positive integer such that \( |f(z)| \leq (1 - |z|)^{-m} \). Prove that

\[
\frac{1}{n!} |f^{(n)}(0)| \leq \frac{(m + n)^{m+n}}{m^m n^n}.
\]

(2) Suppose \( f(z) = Az^2 + Bz + C \). Characterize the coefficients \( A, B, C \in \mathbb{C} \) for which \( f \) is injective on \( \Delta \).

(3) Suppose \( f : \Delta \to \Delta \) is injective and surjective. Show that there exist \( a \in \Delta \) and \( 0 \leq \theta < 2\pi \) such that

\[
f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.
\]

C. (1) Let \( \Delta^* = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \} \). Let \( f : \Delta^* \to \mathbb{C} \) be holomorphic, and suppose there exists \( M \in \mathbb{R} \) such that \( x \leq M \) for all \( z = x + iy \in \Delta^* \). Prove \( a \) is a removable singularity of \( f \). Generalize as in Ahlfors 3.2, problem 5.

(2) Suppose \( f : \mathbb{C} \to \mathbb{C} \) is meromorphic with a pole at \( \infty \) (among possibly others). Show that \( f \) is a quotient of two polynomials.