Math 6350: Homework 3

Due: Friday, October 5

A. Let

$$\Delta_1 = \{z \in \mathbb{C} \mid |z| < 2\}$$
$$\Delta_2 = \left\{z \in \mathbb{C} \mid |z - i| < \frac{1}{2}\right\}.$$

Evaluate the following integrals.

(1) $\displaystyle\int_{\partial\Delta_1} \frac{dz}{z^4 - 1}$,

(2) $\displaystyle\int_{\partial\Delta_2} \frac{dz}{z^4 - 1}$,

(3) $\displaystyle\int_{\partial\Delta_1} z^m (1 - z)^n dz$,

(4) $\displaystyle\int_{\partial\Delta_2} \log(|z|)$.

B. Suppose that f is holomorphic in the open unit disk $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$.

(1) Suppose that there is a positive integer such that $|f(z)| \leq (1 - |z|)^{-m}$. Prove that

$$\frac{1}{n!}|f^{(n)}(0)| \leq \frac{(m+n)^{m+n}}{m^m n^n}.$$

(2) Suppose $f(z) = Az^2 + Bz + C$. Characterize the coefficients $A, B, C \in \mathbb{C}$ for which f is injective on Δ.

(3) Suppose $f : \Delta \to \Delta$ is injective and surjective. Show that there exist $a \in \Delta$ and $0 \leq \theta < 2\pi$ such that

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

C. (1) Let $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Let $f : \Delta^* \to \mathbb{C}$ be holomorphic, and suppose there exists $M \in \mathbb{R}$ such that $x \leq M$ for all $z = x + iy \in \Delta^*$. Prove a is a removable singularity of f. Generalize as in Ahlfors 3.2, problem 5.

(2) Suppose $f : \mathbb{C} \to \mathbb{C}$ is meromorphic with a pole at ∞ (among possibly others). Show that f is a quotient of two polynomials.