Math 6350: Homework 2

Due: Friday, September 21

A. Let $S \subseteq \mathbb{C}$ be an open set, and let $f : S \rightarrow \mathbb{C}$ be a holomorphic function. Prove the following statements.

(1) If $f'(z) = 0$ for all $z \in S$, then $f$ is constant.
(2) If $f(z) \in \mathbb{R}$ for all $z \in S$, then $f$ is constant.
(3) If $z \mapsto f(z)$ is holomorphic, then $f$ is constant.
(4) If $|f(z)|$ is constant, then $f$ is constant.

B. (1) Give a precise definition of a single-valued branch of $\sqrt{1 + z} + \sqrt{1 - z}$, and prove it is holomorphic.
(2) Prove that $f(z)$ and $\overline{f(z)}$ are simultaneously holomorphic.

C. Assume $f$ is holomorphic in an open set $S$ with $f'$ continuous and $|f(z) - 1| < 1$ for $z \in S$. Show

$$\int_{\gamma} \frac{f'(z)}{f(z)} \, dz = 0$$

for every closed curve $\gamma$ in $S$. 