Math 6250: Homework 5

1. (17.2) Prove that if R is local, then every finitely generated projective module is free.

2. (18.3) Let $\mathbb{F}$ be a field, and
$$\mathfrak{b}_n = \{b \in M_n(\mathbb{F}) \mid b_{ji} = 0, 1 \leq i < j \leq n\} \subseteq M_n(\mathbb{F}).$$
Show that ${}_{\mathfrak{b}_n}M_n(\mathbb{F})$ is an injective envelope of ${}_{\mathfrak{b}_n}\mathfrak{b}_n$.

3. (19.7) Let ${}_RM$ and $M^* = \mathrm{Hom}_R(M, R)$.
   (a) Show that $M \otimes_{\mathbb{Z}} M^*$ is a left $R \otimes_{\mathbb{Z}} R^{\mathrm{op}}$-module.
   (b) If ${}_RP$ is finitely generated projective, then ${}_{R\otimes_{\mathbb{Z}}R^{\mathrm{op}}}(P \otimes_{\mathbb{Z}} P^*)$ is finite generated projective.
   (c) If ${}_RG$ is a generator, then ${}_{R\otimes_{\mathbb{Z}}R^{\mathrm{op}}}(G \otimes_{\mathbb{Z}} G^*)$ is a generator.

4. (20.3) Let R, S be rings and $\phi : R \to S$ a homomorphism. Define a functor $T_\phi : {}_S\mathbf{mod} \to {}_R\mathbf{mod}$ by
$$\begin{array}{ccc} R \times T_\phi({}_SM) & \longrightarrow & T_\phi({}_SM) \\ (r, m) & \mapsto & \phi(r)m \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathrm{Hom}_S(M, N) & \longrightarrow & \mathrm{Hom}_R(T_\phi(M), T_\phi(N)) \\ \theta & \mapsto & \begin{array}{rl} T_\phi(\theta) : T_\phi(M) & \to T_\phi(N) \\ m & \mapsto \theta(m). \end{array} \end{array}$$
We may similarly define $T_\phi : \mathbf{mod}_S \to \mathbf{mod}_R$, written as $M_S \mapsto (M_S T_\phi)$.
   (a) Show that the functors $T_\phi(S) \otimes_S \cdot \cong T_\phi \cong \mathrm{Hom}_S((ST_\phi), \cdot)$.
   (b) Show $((ST_\phi) \otimes_R \cdot) \circ T_\phi \cong 1 \cong \mathrm{Hom}_R(T_\phi(S), \cdot) \circ T_\phi$ if ϕ is surjective.

5. Let $G \subseteq H$ be finite groups with a surjective homomorphism $\pi : H \to G$.
   (a) Find ${}_{\mathbb{C}H}W_{\mathbb{C}G} \subseteq \mathbb{C}H$ explicitly so that
   $$\mathrm{Inf}_{\mathbb{C}G}^{\mathbb{C}H}({}_{\mathbb{C}G}N) \cong W \otimes_{\mathbb{C}G} N.$$
   (b) Find ${}_{\mathbb{C}G}M_{\mathbb{C}H} \subseteq \mathbb{C}H$ explicitly so that
   $$\mathrm{Inf}_{\mathbb{C}G}^{\mathbb{C}H}({}_{\mathbb{C}G}N) \cong \mathrm{Hom}_{\mathbb{C}G}(M, N).$$
   (c) Give the isomorphism $W \otimes_{\mathbb{C}G} N \cong \mathrm{Hom}_{\mathbb{C}G}(M, N)$.