Math 6250: Homework 1

1. Consider the symmetric group S_4 and the dihedral group D_8. For each group G
   (a) Give examples of two nonequivalent and nontrivial representations ρ and τ (be sure to show they are not equivalent),
   (b) Construct the corresponding G-modules V_ρ and V_τ,
   (c) Decide whether the modules are reducible,
   (d) Change bases in the module V_ρ and give the new corresponding representation $\rho' : G \to GL_n(\mathbb{C})$.

2. Show that if $\rho : G \to GL(V)$ is a degree one representation, then $G/ker(\rho)$ is an abelian group.

3. Let $\mathrm{GL}_2(\mathbb{F}_q)$ be the general linear group of rank 2 with entries in the field $\mathbb{F}_q$ with q elements. Consider the subalgebra of $\mathbb{C}\mathrm{GL}_2(\mathbb{F}_q)$ given by

$$\mathcal{H}_2(q) = e_B \mathbb{C}\mathrm{GL}_2(\mathbb{F}_q) e_B, \qquad \text{where} \quad e_B = \frac{1}{q} \sum_{\substack{r,s \in \mathbb{F}_q^\times \\ t \in \mathbb{F}_q}} \begin{pmatrix} r & t \\ 0 & s \end{pmatrix}.$$

   (This is the Iwahori-Hecke algebra $\mathcal{H}_2(q)$).
   (a) Find a basis for $\mathcal{H}_2(q)$.
   (b) Give formulas for multiplying basis elements.
   (c) Construct a nontrivial $\mathcal{H}_2(q)$-module that is not the regular module.

4. (2.2 in book)
   (a) Let M be a nonzero abelian group. We have a left action by left endomorphisms $\mathrm{End}^l(M)$ and a right action by right endomorphisms $\mathrm{End}^r(M)$. Show that M is a bimodule if and only if $\mathrm{End}^l(M)$ is commutative.
   (b) Let $\mathbb{C}$ be the complex numbers. Given a $\mathbb{C}$-module V with scalar multiplication $(\alpha, v) \mapsto \alpha v$, we obtain a second $\mathbb{C}$-module structure $\bar{V}$ given by $(\alpha, v) \mapsto \bar{\alpha} v$. Show that neither of these $\mathbb{C}$-modules contains the other, and the two actions do not give a $(\mathbb{C}, \mathbb{C})$-bimodules structure to V.