Math 4450: Homework 9

Due November 4, 2009

Written problems

1. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous on a contour $\Gamma \subseteq \mathbb{C}$. Show that if

$$H(z) = \int_\Gamma \frac{f(w)dw}{(w-z)^2},$$

then

$$H'(z) = 2\int_\Gamma \frac{f(w)dw}{(w-z)^3}.$$

2. Prove the generalized Cauchy integral formula by induction on n. In particular, show that if we assume that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\int_\Gamma \frac{f(w)dw}{(w-z)^n},$$

then

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(w)dw}{(w-z)^{n+1}}.$$

Hint: Write

$$\frac{\int_\Gamma \frac{f(w)dw}{(w-z)^n} - \int_\Gamma \frac{f(w)dw}{(w-a)^n}}{z-a} = \frac{\int_\Gamma \frac{f(w)dw}{(w-z)^{n-1}(w-a)} - \int_\Gamma \frac{f(w)dw}{(w-a)^n}}{z-a} + \int_\Gamma \frac{f(w)dw}{(w-z)^n(w-a)}.$$

For the first term, define a function $h(w) = f(w)/(w-a)$, and use induction. For the last term prove that

$$\lim_{z\to a}\int_\Gamma \frac{f(w)dw}{(w-z)^n(w-a)} = \int_\Gamma \frac{f(w)dw}{(w-a)^{n+1}}.$$

Problems

1. 4.5: 6, 8, 13, 14

2. 4.6: 5, 7