Math 4450: Homework 9

Due November 4, 2009

Written problems

1. Suppose that \( f : \mathbb{C} \to \mathbb{C} \) is continuous on a contour \( \Gamma \subseteq \mathbb{C} \). Show that if

\[
H(z) = \int_{\Gamma} \frac{f(w)dw}{(w-z)^2},
\]

then

\[
H'(z) = 2 \int_{\Gamma} \frac{f(w)dw}{(w-z)^3}.
\]

2. Prove the generalized Cauchy integral formula by induction on \( n \). In particular, show that if we assume that

\[
f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{\Gamma} \frac{f(w)dw}{(w-z)^n},
\]

then

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(w)dw}{(w-z)^{n+1}}.
\]

Hint: Write

\[
\frac{\int_{\Gamma} f(w)dw - \int_{\Gamma} f(w)dw}{(w-z)^n} = \int_{\Gamma} \frac{f(w)dw}{(w-z)^{n+1}} - \int_{\Gamma} \frac{f(w)dw}{(w-a)^n} + \int_{\Gamma} \frac{f(w)dw}{(w-z)^n (w-a)}.
\]

For the first term, define a function \( h(w) = f(w)/(w-a) \), and use induction. For the last term prove that

\[
\lim_{z \to a} \int_{\Gamma} \frac{f(w)dw}{(w-z)^n (w-a)} = \int_{\Gamma} \frac{f(w)dw}{(w-a)^{n+1}}.
\]

Problems

1. 4.5: 6, 8, 13, 14
2. 4.6: 5, 7