Math 4140: Homework 7

Due March 4, 2009

1. For the following graphs,
   (a) Find the corresponding Cartan matrix,
   (b) Find a set $B$ of basis vectors in $\mathbb{R}^6$ such that the graph is the Dynkin diagram of $B$.

2. Suppose $R$ is a root system in $V$. Let
   $$R^\vee = \{ \alpha^\vee \mid \alpha \in R \}.$$  
   (a) Show that $R^\vee$ is a root system of $V$.
   (b) Show that its Cartan matrix is the transpose of the Cartan matrix of $R$.
   (c) Show that $R(B_n)^\vee = R(C_n)$, and $R(D_n)^\vee = R(D_n)$ (see Homework 6).
   (d) Explain why $R$ and $R^\vee$ are not always isomorphic?

   We typically call $R^\vee$ the dual root system to $R$.

3. Let
   $$C_n = \begin{pmatrix}
   2 & -1 & 0 & \cdots & 0 \\
   -1 & 2 & -1 & \ddots & \vdots \\
   0 & \ddots & \ddots & \ddots & 0 \\
   \vdots & \ddots & -1 & 2 & -1 \\
   0 & \cdots & 0 & -1 & 2
   \end{pmatrix}$$

   Show that
   (a) $\det(C_n) = 2 \det(C_{n-1}) - \det(C_{n-2})$ for $n \geq 3$,
   (b) Find a formula for $\det(C_n)$ (a non-recursive formula).

   Hint: Look up the Laplace expansion for the determinant of a matrix.