Math 4140: Homework 6

Due February 25, 2009

1. Let $R$ be a root system with a base $B$.
   (a) Let $\gamma \in R$. Show that the set
   
   $$s_\gamma(B) = \{s_\gamma(\alpha) \mid \alpha \in B\}$$
   
   is also a base for $R$.
   (b) Deduce that if $W$ is the Weyl group of $R$, then $w(B)$ is a base for $R$ for any $w \in W$.

2. Let $W$ be the Weyl group of a root system $R$. Show that for $w \in W$ and $\alpha \in R$,
   
   $$ws_\alpha w^{-1} = s_{w(\alpha)}.$$

3. For each of the following, show that it is an irreducible root system in the vector space spanned by the vectors, and find a base.
   (a) $R(B_n) = \{\pm e_k, \pm (e_i + e_j), \pm (e_i - e_j) \mid 1 \leq k \leq n, 1 \leq i < j \leq n\}$
   (b) $R(C_n) = \{\pm 2e_k, \pm (e_i + e_j), \pm (e_i - e_j) \mid 1 \leq k \leq n, 1 \leq i < j \leq n\}$
   (c) $R(D_n) = \{\pm (e_i + e_j), \pm (e_i - e_j) \mid 1 \leq i < j \leq n\}$

4. For one of the three root systems of Problem 3, do the following.
   (a) For $\alpha, \beta \in B$ (the base you found in Problem 3), find the smallest $m$ such that
   
   $$(s_\alpha s_\beta)^m = 1.$$  
   (b) Do you see any relationship between $m$ and the angle between $\alpha$ and $\beta$? Can you formulate this in a formula for $m$?