Math 4140: Homework 3

Due February 4, 2009

1. Let $\mathbb{F}$ be a field, and let $m, n \in \mathbb{Z}_{\geq 1}$.
   (a) Find the dimension of $M_{m \times n}(\mathbb{F})$ by constructing an explicit basis.
   (b) A *symmetric matrix* is a matrix that is equal to its transpose matrix. Show that the set S of symmetric matrices is a subspace of $M_n(\mathbb{F})$,
   (c) Find the dimension of S by constructing an explicit basis for S.

2. Let $\mathbb{Q}[x]$ be the vector space of polynomials in the variable x with coefficients in $\mathbb{Q}$. Let $f(x) \in \mathbb{Q}[x]$.
   (a) Show that $I = f(x)\mathbb{Q}[x]$ is a subspace of $\mathbb{Q}[x]$.
   (b) Find an explicit basis to find the dimension of the quotient vector space $\mathbb{Q}[x]/f(x)\mathbb{Q}[x]$.

3. Let U and V be subspaces of a finite dimensional vector space W. Define

$$U + V = \{u + v \mid u \in U, v \in V\}.$$

   (a) Show that $U + V$ is a subspace of W.
   (b) Show that

$$\dim(U + V) = \dim(U) + \dim(V) - \dim(U \cap V).$$

   Hint: Construct a basis for $U \cap V$, and supplement it to get bases $\mathcal{B}_U$ and $\mathcal{B}_V$ for U and V, respectively. Show that you now have a basis for $U + V$.
   (c) A *complement* $V \subseteq W$ to a subspace $U \subseteq W$ is a subspace such that $U + V = W$ and $U \cap V = \{0\}$. Show that every subspace of W has a complement.