Math 4140: Homework 10

Due April 15, 2009

1. Let $\mathfrak{g}$ be the Heisenberg algebra.

   (a) Specify a basis for $\mathfrak{g}$,

   (b) Based on this basis compute the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_3(\mathbb{C})$.

   (c) Construct the corresponding $\mathfrak{g}$-module $\mathfrak{g}$.

   (d) Is $\mathfrak{g}$ an irreducible $\mathfrak{g}$-module?

   (e) Find another nonisomorphic, nonzero $\mathfrak{g}$-module, and determine whether it is irreducible.

2. The "natural" representation of a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{gl}_n(\mathbb{C})$ is the representation that sends

$$\rho(h) = h, \qquad \text{for all } h \in \mathfrak{h}.$$

   Let $\mathfrak{b}_n$ be the Lie subalgebra of upper-triangular matrices,

$$\mathfrak{b}_n = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x_{ij} \neq 0, \text{ implies } i \leq j\}.$$

   (a) Construct the $\mathfrak{b}_n$-module V corresponding to the natural representation of $\mathfrak{b}_n$.

   (b) Show that V is not irreducible.

   (c) Show that $\mathfrak{b}_n$ is not semi-simple by finding an irreducible submodule of V that is contained in every submodule of V.

   (d) Explain why (c) shows that $\mathfrak{b}_n$ is not semi-simple.

3. Let

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

   (a) Find a natural basis for $\mathfrak{gl}_3(J, \mathbb{C})$.

   (b) Show that $\mathfrak{gl}_3(J, \mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$. (Therefore $\mathfrak{gl}_3(J, \mathbb{C}) \cong \mathfrak{o}_3(\mathbb{C})$). In fact, using this kind of J for $\mathfrak{o}_n(\mathbb{C})$ will be more useful for us in general.