Math 3170: Homework 4

1. Say a sequence $a_1, a_2, \ldots, a_{2n}$ of n 1's and n -1's is *good* if for each $1 \leq k \leq 2n$, the sum $a_1 + a_2 + \cdots + a_k \geq 0$. Let

$$\text{se}_n = \#\{\text{good sequences of length } 2n\}.$$

   For example,

$$\begin{aligned}\text{se}_3 &= \#\left\{\begin{array}{c}(1,-1,1,-1,1,-1),(1,1,-1,-1,1,-1),(1,-1,1,1,-1,-1),\\(1,1,1,-1,-1,-1),(1,1,-1,1,-1,-1)\end{array}\right\}\\ &= 5.\end{aligned}$$

   Show that se_n is the nth Catalan number by constructing a bijection between Dyck paths and good sequences.

2. Find and prove a closed formula for $S(n, 2)$, $n \geq 2$.

3. Let $k_1, k_2, \ldots, k_\ell$ be positive integers such that $k_1 + k_2 + \cdots + k_\ell = n$. The *multinomial coefficient* $\binom{n}{k_1,k_2,\ldots,k_\ell}$ is the number given by

$$\binom{n}{k_1, k_2, \ldots, k_\ell} = \frac{n!}{k_1! k_2! \cdots k_\ell!}.$$

   (a) Give a description of something that $\binom{n}{k_1,k_2,\ldots,k_\ell}$ counts, and prove your assertion. (In particular, this shows that multinomial coefficients are always integers).

   (b) Give a counting argument to show that

$$\binom{n}{k_1, k_2, \ldots, k_\ell} = \binom{n-1}{k_1 - 1, k_2, \ldots, k_\ell} + \binom{n-1}{k_1, k_2 - 1, \ldots, k_\ell} + \cdots + \binom{n-1}{k_1, k_2, \ldots, k_\ell - 1}.$$

   Note that it might be helpful to review the binomial coefficient recursion.

4. A *set composition* of a set S is a sequence of subsets $(S_1, S_2, \ldots, S_\ell)$ such that

   (1) $S = S_1 \cup S_2 \cup \cdots \cup S_\ell$,

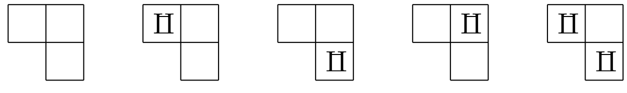
   (2) $S_i \cap S_j = \emptyset$ for $i \neq j$.

   (a) Explain how the set of set partitions of $\{1, 2, \ldots, n\}$ is different from the set of set compositions of $\{1, 2, \ldots, n\}$.

   (b) If C_n is the total number of set compositions of $\{1, 2, \ldots, n\}$, show that

$$C_n = \sum_{k=0}^{n-1} \binom{n}{k} C_k.$$

5. Let r_n be the number of ways to place up to n non-attacking rooks on a triangular chessboard with $n - 1$ boxes on a side. For example, for $n = 3$, we have



   so $r_3 = 5$. Show that $r_n = b_n$ for all n.

   Hint: Number your rows from top to bottom by 1 to $n - 1$, and your columns from left to right by 2 to n, and think about how the coordinates of the rooks might translate into subsets.