Math 3170: Homework 6

Due: October 13, 2010

1. How many 2-digit positive integers are relatively prime to both 2 and 3?

2. For $m \in \mathbb{Z}_{\geq 1}$, let
$$\phi(m) = \#\{j \in \{1, 2, \ldots, m\} \mid \gcd(m, j) = 1\}.$$
Let p, q, r be prime numbers. What is $\phi(pqr)$?

3. For $k \in \mathbb{Z}_{\geq 1}$ compute the coefficients a_n in
$$e^{kx} = \sum_{n \geq 0} a_n \frac{x^n}{n!}$$
in two ways to show that
$$k^n = \sum_{\substack{m_1+m_2+\cdots+m_k=n \\ m_1, m_2, \ldots, m_k \in \mathbb{Z}_{\geq 0}}} \binom{n}{m_1, m_2, \ldots, m_k}.$$

4. Let
$$A(x) = \sum_{n \geq 0} a_n x^n.$$
   (a) Describe the sequence coming from the ordinary generating function
   $$\frac{A(x)}{1-x}.$$
   (b) Describe the sequence coming from the exponential generating function
   $$\frac{A(x)}{1-x}.$$

5. (a) Let
   $$f_{k,n} = \# \left\{ \begin{array}{c} \text{set partitions of } \{1, 2, \ldots, n\} \\ \text{into } k \text{ subsets that contain} \\ \text{at least 2 elements} \end{array} \right\}.$$
   Find and prove a formula for $f_{k,n}$ in terms of the Stirling numbers of the second kind.

   (b) Let
   $$f_n = \# \left\{ \begin{array}{c} \text{set partitions of } \{1, 2, \ldots, n\} \\ \text{into subsets that contain} \\ \text{at least 2 elements} \end{array} \right\}.$$
   Find and prove a formula for f_n in terms of the Bell numbers.