Math 3170: Homework 4

Due: September 22, 2010

1. Find and prove a closed formula for $S(n,2)$, $n \geq 2$.

2. Let $k_1, k_2, \ldots, k_\ell$ be positive integers such that $k_1 + k_2 + \cdots + k_\ell = n$. The *multinomial coefficient* $\binom{n}{k_1,k_2,\ldots,k_\ell}$ is the number given by

$$\binom{n}{k_1, k_2, \ldots, k_\ell} = \frac{n!}{k_1! k_2! \cdots k_\ell!}.$$

   (a) Give a description of something that $\binom{n}{k_1,k_2,\ldots,k_\ell}$ counts, and prove your assertion. (In particular, this shows that multinomial coefficients are always integers).

   (b) Give a counting argument to show that

$$\binom{n}{k_1, k_2, \ldots, k_\ell} = \binom{n-1}{k_1 - 1, k_2, \ldots, k_\ell} + \binom{n-1}{k_1, k_2 - 1, \ldots, k_\ell} + \cdots + \binom{n-1}{k_1, k_2, \ldots, k_\ell - 1}.$$

3. A *set composition* of a set S is a sequence of subsets $(S_1, S_2, \ldots, S_\ell)$ such that

   (1) $S = S_1 \cup S_2 \cup \cdots \cup S_\ell$,

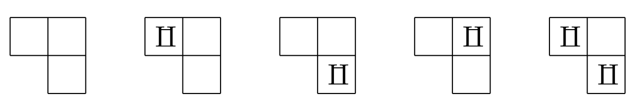
   (2) $S_i \cap S_j = \emptyset$ for $i \neq j$.

   (a) Explain how the set of set partitions of $\{1, 2, \ldots, n\}$ is different from the set of set compositions of $\{1, 2, \ldots, n\}$.

   (b) If C_n is the total number of set compositions of $\{1, 2, \ldots, n\}$, show that

$$C_n = \sum_{k=0}^{n-1} \binom{n}{k} C_k.$$

4. Let r_n be the number of ways to place up to n non-attacking rooks on a triangular chessboard with $n-1$ boxes on a side. For example, for $n = 3$, we have



   so $r_3 = 5$. Show that $r_n = b_n$ for all n.

   Hint: Number your rows from top to bottom by 1 to $n-1$, and your columns from left to right by 2 to n, and think about how the locations of the rooks might translate into subsets.