Math 3140: Homework 3

Due: Wednesday, September 18

A. 5.4, 5.10

B. For D_n, let the vertices of the regular n-gon be labeled by $0, 1, \ldots, n-1$ in a clockwise direction. Let t be symmetry that sends the vertex 0 to $n-1$ and the vertex 1 to $n-2$. Let s be the symmetry that sends 0 to 0 and sends 1 to $n-1$. Show that

$$D_n = \langle s, t \rangle.$$

C. Find examples of the following (justify your answer), or explain why they don't exist:

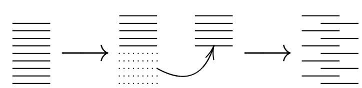
(1) A nonabelian cyclic group.

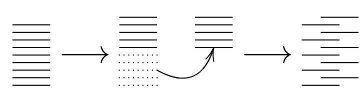
(2) A group generated by two elements.

(3) A group generated by three elements, but not by two element.

D. Show that $(\mathbb{Q}, +)$ is not cyclic. In fact, show that $(\mathbb{Q}, +)$ does not even have a finite number of generators. Hint: What does the subgroup $\langle p/q \rangle$ look like for $p, q \in \mathbb{Z}_{\neq 0}$?

E. A *perfect shuffle* is a shuffle where you split the deck into two equal stacks (the top stack and the bottom stack) and push the stacks together in such a way that the cards from the two stacks alternate. There are two ways of doing this: either the bottom card came from the top stack (an *in-shuffle*), or the bottom card came from the bottom stack (an *out-shuffle*). For example, if you have 10 cards,

Out-shuffle: 

In-shuffle: 

By a judicious use of in and out-shuffles one can create all kinds of card tricks (providing one is capable of a perfect shuffle), since such shuffling is in no way random. Suppose you have a deck of 52 cards.

(1) Prove that if you apply the in-shuffle enough times, you will get back with what you started. Hint: prove that you get back where you started without actually doing it (it's a lot of shuffles).

(2) How many out-shuffles do you need to get back where you started (Justify your answer)?

Remark: It turns out with normal shuffling (which is far from perfect) it takes approximately 7 shuffles to get close to random (which is proved in part using methods of this class).