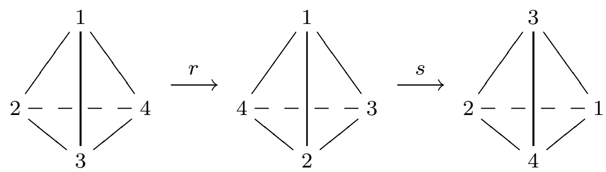
Math 3140: Homework 1

Due: Wednesday, September 5

A. Figure 1.4 essentially describes two symmetries of the tetrahedron:



1.3 Adopt the notation of Figure 1.4. Show that the axis of the composite rotation srs passes through the vertex 4, and that the axis of $rsrr$ is determined by the midpoints of edges 12 and 34.

1.4 Having completed 1.3, Express each of the twelve rotational symmetries of the tetrahedron in terms of r and s.

1.5 Again with the notation of Figure 1.4, check that $r^{-1} = rr$, $s^{-1} = s$, $(rs)^{-1} = srr$, and $(sr)^{-1} = rrs$.

B. 2.5 A function from the plane to itself which preserves the distance between any two points is called and *isometry*. Prove that an isometry must be a bijection and check that the collection of all isometries of the plane forms a group under composition of functions.

2.7 Let x and y be elements of a groups G. Prove that G contains elements w, z which satisfy $wx = y$ and $xz = y$, and show that these elements are unique.

2.8 If x and y are elements of a group, prove that $(xy)^{-1} = y^{-1}x^{-1}$.

C. 3.1 Show that each of the following collections of numbers forms a group under addition.

(i) The even integers.

(ii) All real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

(iii) All real numbers of the form $a + b\sqrt{2}$, where $a, b \in \mathbb{Q}$.

(iv) All complex numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$.

3.5 Let n be a positive integer. Prove that

$$(x \cdot_n y) \cdot_n z = x \cdot_n (y \cdot_n z),$$

for all $x, y, z \in \mathbb{Z}$.

3.9 Let p be a prime number and let x be an integer which satisfies $1 \leq x \leq p - 1$. Show that none of $x, 2x, \ldots, (p-1)x$ is a multiple of p. Deduce the existence of an integer z such that $1 \leq z \leq p - 1$ and $xz(modp) = 1$.

3.10 Use the results of 3.5 and 3.9 to verify that multiplication modulo n makes $\{1, 2, \ldots, n-1\}$ a group if n is prime. What goes wrong if n is not a prime number?

D. What are all the groups with 4 elements? 5 elements?