Math 2001: PHW11

Due: April 13, 2016

1. From the book do:

   11.4. 4,

   12.1. 4, 8

   12.2. 4, 10, 14

2. Let p be a prime number.

   (a) Show that
   $$\binom{p}{j} \equiv 0 (\text{mod } p)$$
   unless $j \in \{0, p\}$.

   (b) Deduce
   $$(x+y)^p \equiv x^p + y^p (\text{mod } p).$$
   Hint: Think binomial theorem.

3. Let R_n be the set of ways to place n non-attacking rooks on an $n \times n$ chess-board.

   (a) Let $f : R_n \to \mathbb{Z}$ be given by
   $$f(r) = \text{number of rooks on the diagonal squares of } r, \qquad \text{for } r \in R_n.$$
   For example, if $n = 4$,
   $$f\left(\begin{array}{|c|c|c|c|}\hline \text{♖} & & & \\ \hline & & & \text{♖} \\ \hline & & \text{♖} & \\ \hline & \text{♖} & & \\ \hline\end{array}\right) = 2, \qquad \text{where} \quad \begin{array}{|c|c|c|c|}\hline \text{♖} & & & \\ \hline & & & \text{♖} \\ \hline & & \text{♖} & \\ \hline & \text{♖} & & \\ \hline\end{array} \in R_4,$$
   and I've shaded the diagonal squares.

   i. What is $f(R_n)$?

   ii. Is f injective?

   iii. Is f surjective?

   iv. Find $|f^{-1}(k)|$ for all $k \in f(R_4)$.

   (b) Find an injective function $g : R_n \to \mathbb{Z}$ (without changing the sets R_n and $\mathbb{Z}$).