One last example of proof by contradiction

Math 2001 class

February 5, 2016

1 Introduction

Proof by contradiction assumes the conclusion is false, and then attempts to show that this contradicts something we already accept to be true. This paper will show that the number of infinite binary sequences is not listable.

We begin with some basic definitions and then will prove the main result in Section 3.

2 Preliminaries

Recall, a *binary sequence* of length n is a sequence $(a_1, a_2, \ldots, a_n)$ where $a_1, a_2, \ldots, a_n \in \{0, 1\}$. An *infinite binary sequence* is a binary sequence where $n = \infty$. Let

$$\{0,1\}^\infty = \{\text{binary sequences of infinite length}\}.$$

3 Main result

Theorem 1. *The elements of* $\{0,1\}^\infty$ *are not listable.*

Proof. We will prove this theorem by contradiction. Assume the conclusion is false: suppose the elements of $\{0,1\}^\infty$ are listable. Let's go ahead and list them.

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & & & & & \end{array}$$

Now mark the diagonal elements red, so the first element in the first sequence, the second element in the second sequences, etc. to get

$$\begin{array}{cccccc}
1 & 0 & 1 & 1 & 0 & \cdots \\
1 & 1 & 0 & 0 & 0 & \cdots \\
0 & 0 & 1 & 1 & 0 & \cdots \\
1 & 0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & 0 & \cdots \\
\vdots & & & & & \ddots
\end{array}$$

Consider the binary sequence obtained by reading only the red elements, and changing each of their values, so

$$0, 0, 0, 1, 1, \ldots$$

Note that this can't be the first sequence, since the 1st entry is different; it can't be the second sequence, since the 2nd entry is different; it can't be the third sequence, since the 3rd entry differs; etc. Thus, the red sequence is an infinite binary sequence that is not in our list. This contradicts that we listed all sequences. □

Remark. *This same method of proof may be used to show that the real numbers* $\mathbb{R}$ *are not listable.*

4 Appendix

One can either install .tex directly on your computer, via tex-live (free), or you can use a site like

https://www.sharelatex.com

There are many things one can do easily with .tex.

$$\sum_{i=1}^{\infty} \frac{1}{n} \qquad \text{diverges.}$$

We can also do nested exponentials like

$$e^{e^{x^2-1} - x^{e^x}} = 0 \tag{4.1}$$

By (4.1), we have that BLAH. Some basic things such as $A \cap B \subseteq A \cup B$. Some numbers are $\leq$ other numbers.

$$\begin{aligned}
1 + 2 + \cdots + n &= (1+n) + (2 + (n-1)) + (3 + (n-2)) + \cdots + \left(\frac{n}{2} + \frac{n-1}{2}\right) \\
&= \underbrace{(n+1) + (n+1) + \cdots + (n+1)}_{\frac{n}{2} \text{ terms}} \\
&= \frac{n(n+1)}{2}.
\end{aligned}$$

4.1 Appendix removed?