Introduction to LaTeX and combinations

Nat Thiem

September 28, 2018

1 Introduction

This document is meant to simultaneously be an introduction to LaTeX and an introduction into the notion of combinations. The word processor LaTeX uses two documents, the .tex file (or source file) and the .pdf file (or output file). We type into the source file and tell the program what to do, and then the program translates it (compiles) into a nicely formatted document. For example, spacing is automatic, and one starts new paragraphs with a double carriage return. Math is done in specific environments. In line math such as $x^2 = 3\pi$ is done with single dollar signs where-as displayed equations use double-dollar signs, such as

$$\sqrt{3x^3 - 4x + 2} = A \cup B.$$

Note that while the output looks good, it may still be nonsense.

Combinations are a way to enumerate subsets of a set of a certain size. In this document, we will define some notation, but not give a formula for the results.

2 Basic Definitions

For $m, n \in \mathbb{Z}_{\geq 0}$, let

$$\binom{n}{m} = |\{A \subseteq \{1, 2, \ldots, n\} \mid |A| = m\}|.$$

We say "n choose m."

Warning. Some of you may know a formula for $\binom{n}{m}$, but it will be invalid until we prove it in this class.

The following theorem gives some basic properties of combinations.

Theorem 2.1. *For $m, n \in \mathbb{Z}_{\geq 0}$,*

(a) We have

$$\binom{n}{m} = \binom{n}{n-m}.$$

(b) We have

$$\binom{n}{0} = \binom{n}{n} = 1.$$

(c) We have

$$\binom{n}{1} = \binom{n}{n-1} = n.$$

(d) We have

$$\sum_{m=0}^{n} \binom{n}{m} = 2^n.$$

Proof. (a) Note that

$$\begin{aligned}
\binom{n}{m} &= |\{A \subseteq \{1, 2, \ldots, n\} \mid |A| = m\}| \\
&= |\{\overline{A} \subseteq \{1, 2, \ldots, n\} \mid |A| = m\}| \\
&= |\{B \subseteq \{1, 2, \ldots, n\} \mid |B| = n - m\}| \\
&= \binom{n}{n-m}.
\end{aligned}$$

(b) By definition,

$$\binom{n}{0} = |\{A \subseteq \{1, 2, \ldots, n\} \mid |A| = 0\}| = |\{\{\}\}| = 1,$$

and $\binom{n}{n} = \binom{n}{n-n} = \binom{n}{0} = 1$ by (a).

(c) By definition,

$$\binom{n}{1} = |\{A \subseteq \{1, 2, \ldots, n\} \mid |A| = 1\}| = |\{\{1\}, \{2\}, \ldots, \{n\}\}| = n,$$

and $\binom{n}{n-1} = \binom{n}{n-(n-1)} = \binom{n}{1} = n$ by (a).

(d) By definition,

$$\begin{aligned}
\sum_{m=0}^{n} \binom{n}{m} &= \sum_{m=0}^{n} |\{A \subseteq \{1, 2, \ldots, n\} \mid |A| = m\}| \\
&= |\{A \subseteq \{1, 2, \ldots, n\}\}| \\
&= |\mathcal{P}(\{1, 2, \ldots, n\})| \\
&= 2^n,
\end{aligned}$$

as desired. □

We can also prove another formula as follows.

Theorem 2.2. *For $n, m \in \mathbb{Z}_{\geq 0}$,*

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}.$$

Proof. We have that

$$\begin{aligned}\{A \subseteq \{1,2,\ldots,n\} \mid |A| = m\} = \{A \subseteq &\{1,2,\ldots,n\} \mid |A| = m, n \in A\} \\ &\cup \{1,2,\ldots,n\} \mid |A| = m, n \notin A\}.\end{aligned}$$

Now,

$$|\{A \subseteq \{1,2,\ldots,n\} \mid |A| = m, n \in A\}| = |\{A \subseteq \{1,2,\ldots,n{-}1\} \mid |A| = m{-}1\}| = \binom{n-1}{m-1},$$

and

$$|\{A \subseteq \{1,2,\ldots,n\} \mid |A| = m, n \notin A\}| = |\{A \subseteq \{1,2,\ldots,n-1\} \mid |A| = m\}| = \binom{n-1}{m}.$$

Putting it together we get

$$\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m},$$

as desired. □

3 Pascal's triangle

$$\begin{array}{ccccccccc}
 & & & & \binom{0}{0} = 1 & & & & \\
 & & & / & & \backslash & & & \\
 & & & \binom{1}{0} = 1 & & \binom{1}{1} = 1 & & & \\
 & & / & & \backslash \quad / & & \backslash & & \\
 & & \binom{2}{0} = 1 & & \binom{2}{1} = 2 & & \binom{2}{2} = 1 & & \\
 & / & & \backslash \quad / & & \backslash \quad / & & \backslash & \\
 & \binom{3}{0} = 1 & & \binom{3}{1} = 3 & & \binom{3}{2} = 3 & & \binom{3}{3} = 1 & \\
 / & & \backslash \quad / & & \backslash \quad / & & \backslash \quad / & & \backslash \\
\binom{4}{0} = 1 & & \binom{4}{1} = 4 & & \binom{4}{2} = 6 & & \binom{4}{3} = 4 & & \binom{4}{4} = 1
\end{array}$$