Math 2001: Homework 7

Due: October 22, 2008

Give complete justifications for all your answers.

Problem 1

1. Note that

\[ 20332 = 2 \cdot 299 \cdot 2 \cdot 17 = 391 \cdot 2 \cdot 13 \cdot 2. \]

Why does this not contradict the unique factorization of numbers into primes?

2. Write addition and multiplication tables for \( \mathbb{Z}_3 \) and \( \mathbb{Z}_4 \).

3. Suppose we wanted to “extend” the concept of divisibility to all integers, including 0. Let us say that an integer \( n \) is divisible by a number \( m \) if there exists an integer \( k \) such that \( n = km \).

   (a) What numbers are divisible by 0?
   (b) What does \( \mathbb{Z}_0 \) look like? How many elements does it have, and what do the congruence classes look like?
   (c) What does \( \mathbb{Z}_{-n} \) look like for negative integers \( -n \)?

Problem 2

1. Which of the following “rules” are true? Either prove or provide a counter-example.

   (a) If \( a \equiv b \pmod{c} \), then \( a + x \equiv b + x \pmod{c + x} \).
   (b) If \( a \equiv b \pmod{c} \), then \( ax \equiv bx \pmod{cx} \).

2. Prove that if \( p \) is prime, and \( 0 < k < p \), then \( p \) divides \( \binom{p}{k} \). Why is \( p \) prime important for this to be true?

   Hint: Use the factorial description of \( \binom{p}{k} \).

3. Use the Binomial Theorem and the previous problem to show that if \( p \) is prime, then

\[ (X + Y)^p \equiv X^p + Y^p \pmod{p}. \]