Math 2001: Homework 14

Due: December 10, 2008

Give complete justifications for all your answers.

Problem 1

1. Let \( n \in \mathbb{Z}_{\geq 1} \). Let
\[
E_n = \{ \text{binary vectors of length } n \text{ with an even number of 1's} \} \\
O_n = \{ \text{binary vectors of length } n \text{ with an odd number of 1's} \}
\]

Show that \(|O_n| = |E_n|\) using a proof by counting and a proof by bijection.

2. Let \( n \in \mathbb{Z}_{\geq 1} \). Prove that
\[
n^3 + (n + 1)^3 + (n + 2)^3
\]
is divisible by 9. Hint: Use induction, and add by 0 for the induction step.

3. Let \( r, n \in \mathbb{Z}_{\geq 0} \). Show that
\[
\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}.
\]
Hint: Fix \( r \) and induct on \( n \).

Problem 2

Prove the following statements by contradiction.

1. Let \( s_1, s_2, \ldots, s_n \in \mathbb{Z} \) be \( n \) integers, and let
\[
a = \frac{s_1 + s_2 + \cdots + s_n}{n}.
\]
Prove that there exists at least one \( i \) such that \( s_i \geq a \).

2. There are no integers \( m, n \in \mathbb{Z} \) such that \( \sqrt{6} = m/n \). Hint: Adapt the book’s proof that \( \sqrt{2} \) is irrational.

Problem 3

Identify whether each of the following statements is true or false. If it is true, prove it. If it is false, then find a counterexample.

1. Let \( A, B, C \) be sets. Then
\[
(A \cap B) \cup C = A \cap (B \cup C).
\]

2. If \( a, b \in \mathbb{Z}_{\geq 1} \) and both \( \sqrt{a} \) and \( \sqrt{b} \) are irrational, then \( \sqrt{ab} \) is irrational.