Math 2001: Homework 14

Due: December 10, 2008

Give complete justifications for all your answers.

Problem 1

1. Let $n \in \mathbb{Z}_{\geq 1}$. Let
$$E_n = \{\text{binary vectors of length } n \text{ with an even number of 1's}\}$$
$$O_n = \{\text{binary vectors of length } n \text{ with an odd number of 1's}\}$$
Show that $|O_n| = |E_n|$ using a proof by counting and a proof by bijection.

2. Let $n \in \mathbb{Z}_{\geq 1}$. Prove that
$$n^3 + (n+1)^3 + (n+2)^3$$
is divisible by 9. Hint: Use induction, and add by 0 for the induction step.

3. Let $r, n \in \mathbb{Z}_{\geq 0}$. Show that
$$\sum_{k=0}^{n} \binom{r+k}{k} = \binom{r+n+1}{n}.$$
Hint: Fix r and induct on n.

Problem 2

Prove the following statements by contradiction.

1. Let $s_1, s_2, \ldots, s_n \in \mathbb{Z}$ be n integers, and let
$$a = \frac{s_1 + s_2 + \cdots + s_n}{n}.$$
Prove that there exists at least one i such that $s_i \geq a$.

2. There are no integers $m, n \in \mathbb{Z}$ such that $\sqrt{6} = m/n$. Hint: Adapt the book's proof that $\sqrt{2}$ is irrational.

Problem 3

Identify whether each of the following statements is true or false. If it is true, prove it. If it is false, then find a counterexample.

1. Let A, B, C be sets. Then
$$(A \cap B) \cup C = A \cap (B \cup C).$$

2. If $a, b \in \mathbb{Z}_{\geq 1}$ and both $\sqrt{a}$ and $\sqrt{b}$ are irrational, then $\sqrt{ab}$ is irrational.