The Combinatorics of Root Systems

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In this lecture, we will assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over the field $\mathbb{C}$ of complex numbers.

Recall from the last lecture that we defined the **lower central series** of $\mathfrak{g}$ to be the sequence of ideals $\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \mathfrak{g}^3 \supset \cdots$ where

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}] \quad \text{for } n \geq 1.$$

- We said that $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}^n = 0$ for some $n \geq 1$.
- We said that $\mathfrak{g}$ is abelian if $\mathfrak{g}^2 = 0$.

We also defined the **derived series** of $\mathfrak{g}$ to be the sequence of subspaces $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$ where

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}^{(n+1)} = \left[\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}\right] \quad \text{for } n \geq 0.$$

- We said that $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(n)} = 0$ for some $n \geq 0$.
- Every nilpotent Lie algebra is solvable.

We defined the **solvable radical** $\mathfrak{r}$ of $\mathfrak{g}$ to be the unique maximal solvable ideal of $\mathfrak{g}$.

- We said that $\mathfrak{g}$ is **semisimple** if $\mathfrak{r} = 0$.
- We said that $\mathfrak{g}$ is **simple** if it contains no ideals other than itself and the zero ideal.
- We called the 1-dimensional Lie algebra the **trivial simple Lie algebra**.
- Every non-trivial simple Lie algebra is semisimple.

We defined a **representation** of $\mathfrak{g}$ of degree n to be a homomorphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathfrak{gl}_n(\mathbb{C})$$

where $\mathfrak{gl}_n(\mathbb{C}) = [M_n(\mathbb{C})]$, the Lie algebra of the associative algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ formed under the commutator product.

We said that a $\mathbb{C}$-vector space V is a $\mathfrak{g}$-**module** if V is equipped with a left $\mathfrak{g}$-action $\mathfrak{g} \times V \to V$ satisfying the following properties:

- $(x, v) \mapsto xv$ is linear in x and v.
- $[x, y]v = x(yv) - y(xv) \quad$ for all $x, y \in \mathfrak{g}$, $v \in V$.

Every n-dimensional $\mathfrak{g}$-module induces a representation of $\mathfrak{g}$ of degree n.

We proved that the vector space $\mathfrak{g}$ forms a $\mathfrak{g}$-module via the action $(x, y) \mapsto [x, y]$.

- We call this module the **adjoint module** and denote the action of x on the vector y by $\operatorname{ad} x \cdot y$.
- We have $\operatorname{ad}[x, y] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x$.

For each 1-dimensional representation $\lambda : \mathfrak{g} \to \mathbb{C}$, we defined the set

$$V_\lambda = \{v \in V \mid (\forall x \in \mathfrak{g})(\exists N(x) \geq 1) \quad (\rho(x) - \lambda(x)1)^{N(x)} v = 0\}.$$

We mentioned that $V = \bigoplus_\lambda V_\lambda$ and that each V_λ is a submodule of V.

If $V_\lambda \neq 0$, then we called λ a **weight** of $\mathfrak{g}$ and V_λ the **weight space** of λ.

For each 1-dimensional representation λ of a nilpotent Lie algebra $\mathfrak{g}$, we can choose a basis of V_λ with respect to which a representation ρ of $\mathfrak{g}$ on V_λ has the form

$$\rho(x) = \begin{pmatrix} \lambda(x) & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda(x) \end{pmatrix} \qquad \text{for all } x \in \mathfrak{g}.$$

We defined the **normalizer** of a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be the set

$$N(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] \in \mathfrak{h} \quad \text{for all } h \in \mathfrak{h}\}.$$

We defined a **Cartan subalgebra** of $\mathfrak{g}$ to be a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is nilpotent and $N(\mathfrak{h}) = \mathfrak{h}$.

We mentioned that if $\mathfrak{h}$ and $\mathfrak{h}'$ are Cartan subalgebras of $\mathfrak{g}$, then $\mathfrak{h}$ and $\mathfrak{h}'$ are conjugate in $\mathfrak{g}$.

By letting a Cartan subalgebra $\mathfrak{h}$ act on $\mathfrak{g}$ via the adjoint action, we obtain a weight space decomposition of $\mathfrak{g}$ of the form $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid (\forall h \in \mathfrak{h})(\exists n \geq 1) \quad (\text{ad } h - \lambda(h)1)^n x = 0\}.$$

We proved that $\mathfrak{h} = \mathfrak{g}_0$.

A 1-dimensional representation λ of $\mathfrak{h}$ is called a **root** of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. We denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by Φ. Thus

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We call this decomposition the **Cartan decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Each $\mathfrak{g}_\alpha$ is called the **root space** of α.

Proposition

If λ and μ are 1-dimensional representations of $\mathfrak{h}$, then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

We defined the **Killing form** of $\mathfrak{g}$ to be the bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $\langle x, y \rangle = \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$.

Proposition

(i) *The Killing form is symmetric, i.e.,* $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in \mathfrak{g}$.

(ii) *The Killing form is invariant, i.e.,* $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$ *for all* $x, y, z \in \mathfrak{g}$.

Proposition

Let $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$ *and let* $x, y \in \mathfrak{a}$. *Then* $\langle x, y \rangle_{\mathfrak{a}} = \langle x, y \rangle_{\mathfrak{g}}$. *Hence the killing form of* $\mathfrak{g}$ *restricted to* $\mathfrak{a}$ *is the Killing form of* $\mathfrak{a}$.

Proposition

If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$.

In particular, we see that $\mathfrak{g}^{\perp}$ is an ideal of $\mathfrak{g}$.

Definition

The Killing form of $\mathfrak{g}$ is **nondegenerate** if $\mathfrak{g}^{\perp} = 0$. The Killing form of $\mathfrak{g}$ is **identically zero** if $\mathfrak{g}^{\perp} = \mathfrak{g}$.

Theorem (Cartan's criterion)

A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate.

From now on, we assume that $\mathfrak{g}$ is a semisimple Lie algebra.

Proposition

If α is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, then $-\alpha$ is also a root.

Theorem

The Cartan subalgebras of $\mathfrak{g}$ are abelian.

Let $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ be the dual space of $\mathfrak{h}$.

We define a map $\mathfrak{h} \to \mathfrak{h}^*$ using the Killing form of $\mathfrak{g}$. Given $h \in \mathfrak{h}$, we define $h^* \in \mathfrak{h}^*$ by

$$h^*(x) = \langle h, x \rangle \qquad \text{for all } x \in \mathfrak{h}.$$

Lemma

The map $h \mapsto h^*$ is an isomorphism of vector spaces between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Notice that Φ is a finite subset of $\mathfrak{h}^*$.

Because the map $h \mapsto h^*$ is bijective, we know that for each $\alpha \in \Phi$, there exists a unique element $h'_\alpha \in \mathfrak{h}$ such that $h'^*_\alpha(x) = \alpha(x)$ for all $x \in \mathfrak{h}$, that is,

$$\alpha(x) = \langle h'_\alpha, x \rangle \qquad \text{for all } x \in \mathfrak{h}.$$

Proposition

The vectors h'_α for $\alpha \in \Phi$ span $\mathfrak{h}$.

Proposition

$h'_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ *for all* $\alpha \in \Phi$.

In fact, we proved that we can choose $e_\alpha \in \mathfrak{g}_\alpha$ and $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $h'_\alpha = [e_\alpha, e_{-\alpha}]$.

Proposition

$\langle h'_\alpha, h'_\alpha \rangle \neq 0$ *for all* $\alpha \in \Phi$.

Theorem

$\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.

Proof.

Choose a 1-dimensional $\mathfrak{h}$-submodule $\mathbb{C}e_\alpha$ of $\mathfrak{g}_\alpha$. We can choose $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h'_\alpha$.

Consider the subspace $\mathfrak{m}$ of $\mathfrak{g}$ given by

$$\mathfrak{m} = \mathbb{C}e_\alpha \oplus \mathbb{C}h'_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} \oplus \cdots .$$

There are only finitely-many summands of $\mathfrak{m}$ since Φ is finite.

We have ad $e_\alpha \cdot \mathfrak{m} \subset \mathfrak{m}$ and ad $e_{-\alpha} \cdot \mathfrak{m} \subset \mathfrak{m}$.

Thus ad $h'_\alpha \cdot \mathfrak{m} = \text{ad}\,[e_\alpha, e_{-\alpha}] \cdot \mathfrak{m} \subset \mathfrak{m}$.

We calculate the trace of ad h'_α on $\mathfrak{m}$ in two different ways. First, we have

$$\begin{aligned}\mathrm{tr}_{\mathfrak{m}}\,(\text{ad}\,h'_\alpha) &= \alpha\,(h'_\alpha) + \dim\mathfrak{g}_{-\alpha}\,(-\alpha\,(h'_\alpha)) + \dim\mathfrak{g}_{-2\alpha}\,(-2\alpha\,(h'_\alpha)) + \cdots \\ &= \alpha\,(h'_\alpha)\,(1 - \dim\mathfrak{g}_{-\alpha} - 2\dim\mathfrak{g}_{-2\alpha} - \cdots).\end{aligned}$$

Second, we have

$$\mathrm{tr}_{\mathfrak{m}}\,(h'_\alpha) = \mathrm{tr}_{\mathfrak{m}}(\text{ad}\,e_\alpha\ \text{ad}\,e_{-\alpha}) - \mathrm{tr}_{\mathfrak{m}}(\text{ad}\,e_{-\alpha}\ \text{ad}\,e_\alpha) = 0.$$

Thus

$$\alpha\,(h'_\alpha)\,(1 - \dim\mathfrak{g}_{-\alpha} - 2\dim\mathfrak{g}_{-2\alpha} - \cdots) = 0.$$

Now $\alpha(h'_\alpha) = \langle h'_\alpha, h'_\alpha \rangle \neq 0$, and so

$$1 - \dim \mathfrak{g}_{-\alpha} - 2 \dim \mathfrak{g}_{-2\alpha} - \cdots = 0.$$

This can happen only if $\dim \mathfrak{g}_{-\alpha} = 1$ and $\dim \mathfrak{g}_{-r\alpha} = 0$ for all $r \geq 2$.

Now $\alpha \in \Phi$ if and only if $-\alpha \in \Phi$. Thus $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. □

Proposition

If $\alpha \in \Phi$ and $r\alpha \in \Phi$ where $r \in \mathbb{Z}$, then $r = 1$ or $r = -1$.

Proof.

From the above, we have $\dim \mathfrak{g}_{-r\alpha} = 0$ for all $r \geq 2$, that is, $-r\alpha$ is not a root.

Now $r\alpha \in \Phi$ if and only if $-r\alpha \in \Phi$. Thus only α and $-\alpha$ can be roots.

□

Let $\alpha, \beta \in \Phi$ be roots such that $\beta \neq \alpha$ and $\beta \neq -\alpha$. Then β is not an integer multiple of α.

There do, however, exist integers $p \geq 0, q \geq 0$ such that the elements

$$-p\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

all lie in Φ but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ do not.

The set of roots

$$-p\alpha + \beta, \ldots, q\alpha + \beta$$

is called the $\boldsymbol{\alpha}$**-chain** of roots through β.

Proposition

Let α, β be roots such that $\beta \neq \alpha$ and $\beta \neq -\alpha$. Let

$$-p\alpha + \beta, \dots, \beta, \dots, q\alpha + \beta$$

be the α-chain of roots through β. Then

$$\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q.$$

Proof.

Consider the subspace $\mathfrak{m}$ of $\mathfrak{g}$ given by

$$\mathfrak{m} = \mathfrak{g}_{-p\alpha+\beta} \oplus \cdots \oplus \mathfrak{g}_{q\alpha+\beta}.$$

Recall that $h'_\alpha = [e_\alpha, e_{-\alpha}] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Now β is not an integer multiple of α, and so $-(p+1)\alpha+\beta \neq 0$ and $(q+1)\alpha+\beta \neq 0$.

We have $\operatorname{ad} e_\alpha \cdot \mathfrak{g}_{q\alpha+\beta} \subset \mathfrak{g}_{(q+1)\alpha+\beta}$. Because $(q+1)\alpha+\beta \neq 0$ and $(q+1)\alpha+\beta \notin \Phi$, we must have $\mathfrak{g}_{(q+1)\alpha+\beta} = 0$.

Thus $\operatorname{ad} e_\alpha \cdot \mathfrak{m} \subset \mathfrak{m}$. By a similar argument, we have $\operatorname{ad} e_{-\alpha} \cdot \mathfrak{m} \subset \mathfrak{m}$, and so

$$\operatorname{ad} h'_\alpha \cdot \mathfrak{m} = (\operatorname{ad} e_\alpha \ \operatorname{ad} e_{-\alpha} - \operatorname{ad} e_{-\alpha} \ \operatorname{ad} e_\alpha)\mathfrak{m} \subset \mathfrak{m}.$$

We calculate the trace of $\operatorname{ad} h'_\alpha$ on $\mathfrak{m}$ in two different ways. We have

$$\operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} h'_\alpha) = \sum_{i=-p}^{q} (i\alpha+\beta)(h'_\alpha)$$

since $\dim \mathfrak{g}_{i\alpha+\beta} = 1$.

Second, we have

$$\mathrm{tr}_{\mathfrak{m}}\,(\mathrm{ad}\; h'_\alpha) = \mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\; e_\alpha\;\; \mathrm{ad}\; e_{-\alpha}) - \mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\; e_{-\alpha}\;\; \mathrm{ad}\; e_\alpha) = 0.$$

Thus

$$\sum_{i=-p}^{q} (i\alpha + \beta)\,(h'_\alpha) = 0,$$

that is,

$$\left(\frac{q(q+1)}{2} - \frac{p(p+1)}{2}\right)\alpha\,(h'_\alpha) + (p+q+1)\beta\,(h'_\alpha) = 0.$$

Since $p + q + 1 \neq 0$, this yields

$$\frac{(q-p)}{2}\left\langle h'_\alpha, h'_\alpha\right\rangle + \left\langle h'_\alpha, h'_\beta\right\rangle = 0.$$

Hence

$$\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q$$

since $\langle h'_\alpha, h'_\alpha \rangle \neq 0$. □

Corollary

If $\alpha \in \Phi$ and $\zeta\alpha \in \Phi$ where $\zeta \in \mathbb{C}$, then $\zeta = 1$ or $\zeta = -1$.

Proof.

Suppose $\zeta \neq \pm 1$ and let $\beta = \zeta\alpha$. Then $\beta(h'_\alpha) = \zeta\alpha(h'_\alpha)$, that is,

$$\langle h'_\alpha, h'_\beta \rangle = \zeta \langle h'_\alpha, h'_\alpha \rangle.$$

From the previous proposition, this yields

$$2\zeta = 2\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q.$$

Hence $2\zeta \in \mathbb{Z}$. If $\zeta \in \mathbb{Z}$, then $\zeta = \pm 1$. Thus $\zeta \notin \mathbb{Z}$. It follows that $p - q$ is odd.

The α-chain of roots through β is

$$-\left(\frac{p+q}{2}\right)\alpha, \ldots, \beta = \left(\frac{p-q}{2}\right)\alpha, \ldots, \left(\frac{p+q}{2}\right)\alpha.$$

Since $p - q$ is odd and consecutive roots differ by α, we see that all roots in the α-chain are odd multiples of $\frac{1}{2}\alpha$.

Also, $p - q \neq 0$, and so p and q cannot both be zero. Thus $p + q \neq 0$.

Because the first and last roots are negatives of one another, $\frac{1}{2}\alpha$ must lie in the α-chain. Thus $\frac{1}{2}\alpha \in \Phi$.

But $\alpha \in \Phi$, and so $2\left(\frac{1}{2}\alpha\right) \in \Phi$, a contradiction. $\square$

Proposition

$\langle h'_\alpha, h'_\beta \rangle \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Phi$.

Proof.

We already know that $\langle h'_\alpha, h'_\beta \rangle \in \mathbb{C}$. We also have

$$2\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} \in \mathbb{Z}.$$

Thus $\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} \in \mathbb{Q}$. It is therefore sufficient to show that $\langle h'_\alpha, h'_\alpha \rangle \in \mathbb{Q}$.

We have

$$\langle h'_\alpha, h'_\alpha \rangle = \mathrm{tr}\left(\mathrm{ad}\ h'_\alpha\ \mathrm{ad}\ h'_\alpha\right) = \sum_{\beta\in\Phi} \left(\beta\left(h'_\alpha\right)\right)^2 = \sum_{\beta\in\Phi} \langle h'_\alpha, h'_\beta \rangle^2.$$

Dividing by $\langle h'_\alpha, h'_\alpha \rangle^2$, this yields

$$\frac{1}{\langle h'_\alpha, h'_\alpha \rangle} = \sum_{\beta \in \Phi} \left(\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} \right)^2 \in \mathbb{Z}.$$

Hence $\langle h'_\alpha, h'_\alpha \rangle \in \mathbb{Q}$, completing the proof. □

Recall that the vectors h'_α for $\alpha \in \Phi$ span $\mathfrak{h}$. We choose a subset $h'_{\alpha_1}, \ldots, h'_{\alpha_\ell}$ that forms a basis of $\mathfrak{h}$.

Proposition

If $\alpha \in \Phi$, then $h'_\alpha = \sum_{i=1}^{\ell} \mu_i h'_{\alpha_i}$ where each μ_i lies in $\mathbb{Q}$.

Proof.

Let $\langle h'_{\alpha_i}, h'_{\alpha_j} \rangle = \xi_{ij}$. Then $\xi_{ij} \in \mathbb{Q}$.

Consider the system of equations

$$\begin{aligned}
\langle h'_\alpha, h'_{\alpha_1} \rangle &= \mu_1 \xi_{11} + \mu_2 \xi_{21} + \cdots + \mu_\ell \xi_{\ell 1} \\
\langle h'_\alpha, h'_{\alpha_2} \rangle &= \mu_1 \xi_{12} + \mu_2 \xi_{22} + \cdots + \mu_\ell \xi_{\ell 2} \\
&\vdots \\
\langle h'_\alpha, h'_{\alpha_\ell} \rangle &= \mu_1 \xi_{1\ell} + \mu_2 \xi_{2\ell} + \cdots + \mu_\ell \xi_{\ell\ell}.
\end{aligned}$$

Since the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate, we have $\det(\xi_{ij}) \neq 0$.

Thus we may solve this system of equations for $\mu_1, \ldots, \mu_\ell$ in terms of the $\langle h'_\alpha, h'_{\alpha_i} \rangle$ and ξ_{ij}. □

We define $\mathfrak{h}_{\mathbb{Q}}$ to be the set of all elements of the form $\sum_{i=1}^{\ell} \mu_i h'_{\alpha_i}$ with $\mu_i \in \mathbb{Q}$. Similarly, we define $\mathfrak{h}_{\mathbb{R}}$ to be the set of all such elements with $\mu_i \in \mathbb{R}$.

Proposition

Let $x \in \mathfrak{h}_{\mathbb{R}}$. Then $\langle x, x \rangle \in \mathbb{R}$ and $\langle x, x \rangle \geq 0$. If $\langle x, x \rangle = 0$, then $x = 0$.

Proof.

Write $x = \sum_{i=1}^{\ell} \mu_i h'_{\alpha_i}$ where $\mu_i \in \mathbb{R}$.

One can check that $\langle x, x \rangle = \sum_{\lambda \in \Phi} \left(\sum_i \mu_i \lambda \left(h'_{\alpha_i} \right) \right)^2$.

Now $\lambda \left(h'_{\alpha_i} \right) = \left\langle h'_{\lambda}, h'_{\alpha_i} \right\rangle \in \mathbb{Q}$. Thus $\langle x, x \rangle \in \mathbb{R}$, and also $\langle x, x \rangle \geq 0$.

If $\langle x, x \rangle = 0$, then $\sum_i \mu_i \lambda \left(h'_{\alpha_i} \right) = 0$ for all $\lambda \in \Phi$. This yields $\sum_i \mu_i \left\langle h'_{\alpha_i}, h'_{\alpha_j} \right\rangle = 0$, that is, $\sum_i \mu_i \xi_{ij} = 0$.

Since the matrix (ξ_{ij}) is non-singular, we must have $\mu_i = 0$ for all i. $\square$

This the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}_{\mathbb{R}}$ is a map

$$\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}} \to \mathbb{R}$$

that is a symmetric positive definite bilinear form.

The vector space $\mathfrak{h}_{\mathbb{R}}$ endowed with this positive definite form is thus a Euclidean space containing all vectors h'_α for $\alpha \in \Phi$.

Recall that the map $h \mapsto h^*$ given by $h^*(x) = \langle h, x \rangle$ is an isomorphism from $\mathfrak{h}$ into $\mathfrak{h}^*$.

We define $\mathfrak{h}^*_{\mathbb{R}}$ to be the image of $\mathfrak{h}_{\mathbb{R}}$ under this isomorphism. Thus $\mathfrak{h}^*_{\mathbb{R}}$ is the real subspace of $\mathfrak{h}^*$ spanned by Φ.

We may also define a symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$ by

$$\left\langle h_1^*, h_2^* \right\rangle = \langle h_1, h_2 \rangle \in \mathbb{R}.$$

Thus $\mathfrak{h}_{\mathbb{R}}^*$ becomes a Euclidean space containing all roots $\alpha \in \Phi$. For convenience, we write $V = \mathfrak{h}_{\mathbb{R}}^*$.

Definition

A **total ordering** on V is a relation $\prec$ satisfying the following conditions:

(i) If $\lambda \prec \mu$ and $\mu \prec \nu$, then $\lambda \prec \nu$.

(ii) If $\lambda, \mu \in V$, then exactly one of the following holds: $\lambda \prec \mu$, $\lambda = \mu$, or $\mu \prec \lambda$.

(iii) If $\lambda \prec \mu$, then $\lambda + \nu \prec \mu + \nu$.

(iv) If $\lambda \prec \mu$ and $\xi \in \mathbb{R}$ with $\xi > 0$, then $\xi\lambda \prec \xi\mu$.

Definition

A **positive system** $\Phi^+ \subset \Phi$ is the set of all roots $\alpha \in \Phi$ satisfying $0 \prec \alpha$ for some total ordering on V. Given a positive system Φ^+, we define a **fundamental system** $\Pi \subset \Phi^+$ by

$$\Pi = \{\alpha \in \Phi^+ \mid \alpha \text{ cannot be expressed as the sum of two elements of } \Phi^+\}.$$

We define Φ^- to be the corresponding set of negative roots.

Proposition

Every root in Φ^+ is a sum of roots in Π.

Proof.

If $\alpha \in \Phi^+ \backslash \Pi$, then $\alpha = \beta + \gamma$ where $\beta, \gamma \in \Phi^+$ and $\beta \prec \alpha, \gamma \prec \alpha$. □

Proposition

If $\alpha, \beta \in \Pi$ *and* $\alpha \neq \beta$*, then* $\langle \alpha, \beta \rangle \leq 0$*.*

Proof.

We cannot have $\alpha - \beta \in \Phi$, for otherwise $\alpha - \beta \in \Phi^+$ or $\beta - \alpha \in \Phi^+$.

This would yield $\alpha = \beta + (\alpha - \beta)$ or $\beta = \alpha + (\beta - \alpha)$, a contradiction.

Thus the α-chain of roots through β is of the form

$$\beta, \alpha + \beta, \ldots, q\alpha + \beta$$

We have

$$2\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\left\langle h'_\alpha, h'_\alpha \right\rangle} = -q.$$

We proved before that $\left\langle h'_\alpha, h'_\alpha \right\rangle > 0$. Thus $\left\langle h'_\alpha, h'_\beta \right\rangle \leq 0$. □

Theorem

A fundamental system Π *forms a basis of* $V = \mathfrak{h}^*_{\mathbb{R}}$*.*

Proof.

We know that Φ spans V. Since $\alpha \in \Phi$ if and only if $-\alpha \in \Phi$, we see that Φ^+ spans V. Thus Π spans V.

Now suppose Π is not linearly independent. Then there exists a non-trivial linear combination of roots $\alpha_i \in \Pi$ that is equal to zero.

Taking the terms with positive coefficients to one side of the equation, we obtain

$$v = \mu_{i_1}\alpha_{i_1} + \cdots + \mu_{i_r}\alpha_{i_r} = \mu_{j_1}\alpha_{j_1} + \cdots + \mu_{j_r}\alpha_{j_r}$$

where $\mu_{i_1}, \ldots, \mu_{i_r}, \mu_{j_1}, \ldots, \mu_{j_s} > 0$.

We have $\langle v, v\rangle = \langle \mu_{i_1}\alpha_{i_1} + \cdots + \mu_{i_r}\alpha_{i_r}, \mu_{j_1}\alpha_{j_1} + \cdots + \mu_{j_r}\alpha_{j_r}\rangle$.

Expanding this linearly, we see that $\langle v, v\rangle \leq 0$. Since the bilinear form is positive definite, this implies $v = 0$.

But $0 \prec v$ since we have $0 \prec \alpha_i$ and $\mu_i > 0$. □

It follows that $|\Pi| = \ell = \dim \mathfrak{h}$, that is, the number of roots in a fundamental system is equal to the rank of the Lie algebra $\mathfrak{g}$.

Corollary

Let Π be a fundamental system of roots. Then each $\alpha \in \Phi$ can be expressed in the form $\alpha = \sum n_i\alpha_i$ where $\alpha_i \in \Pi$, $n_i \in \mathbb{Z}$, and $n_i \geq 0$ for all i or $n_i \leq 0$ for all i.

For each $\alpha \in V$, we define a linear map $s_\alpha : V \to V$ by

$$s_\alpha(x) = x - 2\frac{\langle \alpha, x \rangle}{\langle \alpha, \alpha \rangle}\alpha \qquad \text{for all } x \in V$$

where, as before, $V = \mathfrak{h}^*_{\mathbb{R}}$. One sees immediately that s_α satisfies

$$s_\alpha(\alpha) = -\alpha$$

$$s_\alpha(x) = x \qquad \text{if } \langle \alpha, x \rangle = 0.$$

Thus s_α is the reflection in the hyperplane of V orthogonal to α.

Definition

The group W of all non-singular maps on V generated by the s_α for all $\alpha \in \Phi$ is called the **Weyl group**.

Proposition

The Weyl group is a group of isometries on V, that is,

$$\langle w(x), w(y)\rangle = \langle x, y\rangle \qquad \text{for all } x, y \in V,\ w \in W.$$

Proposition

If $\alpha \in \Phi$ and $w \in W$, then $w(\alpha) \in \Phi$, i.e., W permutes the roots.

Proof.

It suffices to show that $s_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in \Phi$.

The statement is clear if β is either α or $-\alpha$. So suppose $\beta \neq \pm\alpha$ and let

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$$

be the α-chain of roots through β.

Then we have

$$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha = \beta - (p-q)\alpha.$$

This element is in the α-chain. □

Corollary

The Weyl group W is finite.

Lemma

Let $\alpha \in \Pi$. If $\beta \in \Phi^+$ and $\beta \neq \alpha$, then $s_\alpha(\beta) \in \Phi^+$.

Proof.

We can express β in the form

$$\beta = \sum_i n_i \alpha_i \qquad \alpha_i \in \Pi, \qquad n_i \in \mathbb{Z}, \qquad n_i \geq 0.$$

Since $\beta \neq \alpha$, there must be some $n_i \neq 0$ with $\alpha_i \neq \alpha$. We have

$$s_\alpha(\beta) = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

We express $s_\alpha(\beta)$ as a linear combination of the elements of Π. Since $\beta \neq 0$, we have $s_\alpha(\beta) \in \Phi^+$ or $s_\alpha(\beta) \in \Phi^-$.

If $s_\alpha(\beta) \in \Phi^+$, then the integer coefficients of the elements of Π will all be nonnegative, and vice-versa.

Since the coefficient of α_i remains n_i, we must have $s_\alpha(\beta) \in \Phi^+$. $\square$

Theorem

If Φ_1^+, Φ_2^+ are two positive systems in Φ, then there exists a $w \in W$ such that $w\left(\Phi_1^+\right) = \Phi_2^+$.

Proof.

Let $m = \left|\Phi_1^+ \cap \Phi_2^-\right|$. The proof is by induction on m.

If $m = 0$, then $\Phi_1^+ = \Phi_2^+$, and so $w = 1$ satisfies the theorem.

Now assume $m > 0$. Then $\Phi_1^+ \cap \Phi_2^- \neq \varnothing$. We cannot have $\Pi_1 \subset \Phi_2^+$, as this would imply $\Phi_1^+ \subset \Phi_2^+$, contrary to $m > 0$.

Thus there exists an $\alpha \in \Pi_1 \cap \Phi_2^-$. Consider the positive system $s_\alpha\left(\Phi_1^+\right)$ in Φ. By the lemma, $s_\alpha\left(\Phi_1^+\right)$ contains all roots in Φ_1^+ except α, together with $-\alpha$.

It follows that

$$\left| s_\alpha \left(\Phi_1^+ \right) \cap \Phi_2^- \right| = m - 1.$$

By induction, there exists a $w' \in W$ such that $w' s_\alpha \left(\Phi_1^+ \right) = \Phi_2^+$.

Letting $w = w' s_\alpha$, we have $w \left(\Phi_1^+ \right) = \Phi_2^+$ as desired. □

Corollary

If Π_1, Π_2 are two fundamental systems in Φ, then there exists a $w \in W$ such that $w(\Pi_1) = \Pi_2$.

Proposition

Let $\alpha, \beta \in \Phi$ *such that* $\beta \neq \pm\alpha$. *Then*

(i) *The angle between* α, β *is one of*

$\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$.

(ii) *If* α, β *are inclined at* $\pi/3$ *or* $2\pi/3$, *then* α, β *have the same length.*

(iii) *If* α, β *are inclined at* $\pi/4$ *or* $3\pi/4$, *then the ratio of their lengths is* $\sqrt{2}$.

(iv) *If* α, β *are inclined at* $\pi/6$ *or* $5\pi/6$, *then the ratio of their lengths is* $\sqrt{3}$.

Proof.

We have $\langle \alpha, \beta \rangle = |\alpha||\beta| \cos\theta$.

Thus $\cos^2\theta = \frac{\langle \alpha,\beta \rangle^2}{\langle \alpha,\alpha \rangle \langle \beta,\beta \rangle} = \frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \cdot \frac{\langle \beta,\alpha \rangle}{\langle \beta,\beta \rangle}$.

Hence $4\cos^2\theta = 2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle} \cdot 2\frac{\langle \beta,\alpha \rangle}{\langle \beta,\beta \rangle}$.

Recall that $2\frac{\langle \alpha,\beta \rangle}{\langle \alpha,\alpha \rangle}$ and $2\frac{\langle \beta,\alpha \rangle}{\langle \beta,\beta \rangle}$ are integers. Hence $4\cos^2\theta \in \mathbb{Z}$.

Since $0 \leq 4\cos^2\theta \leq 4$ and $\beta \neq \pm\alpha$, we must have $4\cos^2\theta \in \{0, 1, 2, 3\}$.

In each case, we consider the possible factorization of $4\cos^2\theta$ into the product of two integers. □

Corollary

Let Π *be a fundamental system of roots in* Φ *and let* $\alpha, \beta \in \Pi$ *with* $\beta \neq -\alpha$. *Then the angle between* α, β *is one of* $\pi/2$, $2\pi/3$, $3\pi/4$, $5\pi/6$.

Now let $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ be a fundamental system. For $i, j = 1, \ldots, \ell$, we define the elements

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}.$$

We have $A_{ij} \in \mathbb{Z}$ for all i, j.

Definition

The $\ell \times \ell$ matrix $A = (A_{ij})$ is called the **Cartan matrix**.

Proposition

The Cartan matrix A has the following properties:

(i) $A_{ii} = 2$ *for all* i.

(ii) $A_{ij} \in \{0, -1, -2, -3\}$ *if* $i \neq j$.

(iii) *If* $A_{ij} = -2$ *or* -3, *then* $A_{ji} = -1$.

(iv) $A_{ij} = 0$ *if and only if* $A_{ji} = 0$.

Proposition

The Cartan matrix of $\mathfrak{g}$ *depends only on the indexing of the fundamental roots. It is independent of the choice of Cartan subalgebra* $\mathfrak{h}$ *and fundamental system* Π.

Proof.

The independence of the choice of Cartan subalgebra follows from the conjugacy of Cartan subalgebras.

Let Π' be another fundamental system. Recall that $w(\Pi) = \Pi'$ for some $w \in W$.

For each i, we write $w(\alpha_i) = \alpha'_i$. Because w is an isometry of V, we have

$$2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = 2\frac{\langle \alpha'_i, \alpha'_j \rangle}{\langle \alpha'_i, \alpha'_i \rangle}.$$

Thus $A_{ij} = A'_{ij}$ for all $i, j = 1, \ldots, \ell$. □

Example

We now determine the Cartan matrices for $\ell = 1$ and 2 using Proposition. The only possible 1×1 Cartan matrix is (2). We also see that any 2×2 Cartan matrix must be one of the following:

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

Definition

The **Dynkin diagram** is a graph with vertices labeled $1, \ldots, \ell$ such that for any two distinct indices i, j, the number of edges joining i, j is n_{ij} where

$$n_{ij} = A_{ij}A_{ji}.$$

Notice that the Dynkin diagram is uniquely determined by the semisimple Lie algebra $\mathfrak{g}$.

Proposition

$n_{ij} \in \{0, 1, 2, 3\}$ for all $i \neq j$.

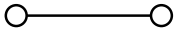
Example

The Dynkin diagrams of the Cartan matrices for $\ell = 1$ and 2 are

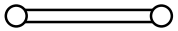
$$(2)$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

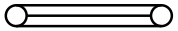
$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$$



$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$



If we renumber the vertices of the Dynkin diagram so that those in each connected component are numbered consecutively, then the Cartan matrix will split into blocks of the form

$$A = \left(\begin{array}{c|c|c|c} * & 0 & 0 & 0 \\ \hline 0 & * & 0 & 0 \\ \hline 0 & 0 & * & 0 \\ \hline 0 & 0 & 0 & * \end{array}\right)$$

where each block corresponds to a connected component.

- Each block will, in fact, be the Cartan matrix for its corresponding connected component.
- The set $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ will be partitioned into subsets in a corresponding way.
- Roots in different subsets will be mutually orthogonal.

It turns out that the number of possible connected Dynkin diagrams is quite limited.

In order to reduce the list of possibilities, it is useful to introduce a quadratic form $Q(x_1, \ldots, x_\ell)$ defined by

$$Q(x_1, \ldots, x_\ell) = 2\sum_{i=1}^{\ell} x_i^2 - \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \sqrt{n_{ij}}\, x_i x_j.$$

Proposition

The quadratic form $Q(x_1, \ldots, x_\ell)$ is positive definite.

Proof.

Using the definition of n_{ij}, we obtain

$$Q(x_1, \ldots, x_\ell) = \sum_{i,j=1}^{\ell} 2\frac{\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|} x_i x_j = 2\left\langle \sum_{i=1}^{\ell} \frac{x_i \alpha_i}{|\alpha_i|}, \sum_{j=1}^{\ell} \frac{x_j \alpha_j}{|\alpha_j|} \right\rangle = 2\langle y, y \rangle$$

where $y = \sum_{i=1}^{\ell} \frac{x_i \alpha_i}{|\alpha_i|}$.

Thus $Q(x_1, \ldots, x_\ell) \geq 0$ since the bilinear form $\langle , \rangle$ is positive definite.

Furthermore, if $Q(x_1, \ldots, x_\ell) = 0$, then $y = 0$. In this case, since $\alpha_1, \ldots, \alpha_\ell$ are linearly independent, we must have $x_i = 0$ for all i. □

Thus we have shown that the the connected components of the Dynkin diagram of a semisimple Lie algebra have the following properties:

(A) The graph is connected.

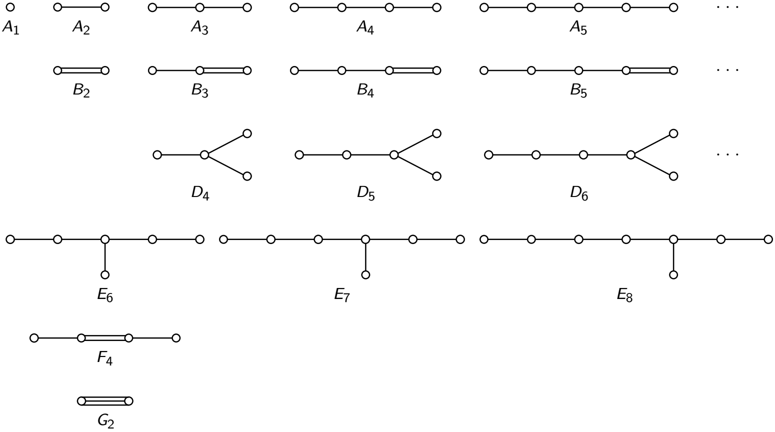
(B) Any pair of distinct vertices are joined by 0, 1, 2, or 3 edges.

(C) The corresponding quadratic form $Q(x_1, \ldots, x_\ell)$ is positive definite.

In order to find all possible Dynkin diagrams, we are going to to determine all graphs satisfying conditions (A), (B), (C).

Theorem

The graphs satisfying conditions (A), (B), (C) are precisely those in the following list:



Proof.

These graphs clearly satisfy conditions (A) and (B).

A quadratic form $\sum a_{ij}x_ix_j$ is positive definite if and only if the leading minors of its symmetric matrix (a_{ij}) have positive determinant, that is,

$$|a_{11}| > 0, \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0, \quad \ldots, \quad \det(a_{ij}) > 0.$$

Given a graph Γ on the list with ℓ vertices, we show that $Q(x_1, \ldots, x_\ell)$ is positive definite by induction on ℓ.

If $\ell = 1$, then $\Gamma = A_1$ and $Q(x_1) = 2x_1^2$ is positive definite.

If $\ell = 2$, then $\Gamma = A_2$, B_2, or G_2. In each case, the symmetric matrix corresponding to $Q(x_1, x_2)$ is

$$A_2 : \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad B_2 : \begin{pmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{pmatrix} \qquad G_2 : \begin{pmatrix} 2 & -\sqrt{3} \\ -\sqrt{3} & 2 \end{pmatrix}.$$

The leading minors of these matrices have positive determinant.

Now suppose $\ell \geq 3$. Looking at the list, we see that Γ contains at least one vertex that is joined to just one other vertex, and joined to it by a single edge.

Label such a vertex ℓ and label the vertex it is joined to $\ell - 1$.

We write $\Gamma = \Gamma_\ell$. We denote the graph obtained from Γ_ℓ by removing the vertex ℓ by $\Gamma_{\ell-1}$ and the graph obtained from $\Gamma_{\ell-1}$ by removing the vertex $\ell - 1$ by $\Gamma_{\ell-2}$.

Observe that $\Gamma_{\ell-1}$ and $\Gamma_{\ell-2}$ are also on the list.

Let $\det \Gamma_\ell$ be the determinant of the symmetric matrix representing the quadratic form $Q(x_1, \ldots, x_\ell)$ associated with Γ_ℓ. We obtain the equality

$$\det \Gamma_\ell = \begin{vmatrix} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & 2 & -1 \\ 0 & \ldots & 0 & -1 & 2 \end{vmatrix} = 2 \det \Gamma_{\ell-1} - \det \Gamma_{\ell-2}$$

by expanding the determinant by its last row.

This gives an inductive way to calculate $\det \Gamma_\ell$. We perform this calculation for each Γ_ℓ on the list.

Lemma

Let Γ be a graph satisfying conditions (A), (B), (C). Let Γ' be a connected subgraph of Γ. Then Γ' satisfies conditions (A), (B), (C), also.

Proof.

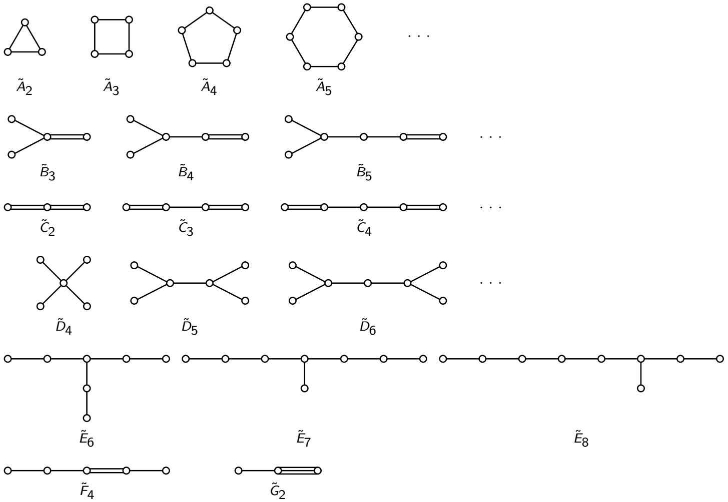
Let $Q(x_1, \ldots, x_\ell)$ be the quadratic form of Γ and $Q'(x_1, \ldots, x_m)$ the quadratic form of Γ' where $m \leq \ell$.

Suppose Q' is not positive definite. Then there exist $y_1, \ldots, y_m \in \mathbb{R}$, not all zero, such that $Q'(y_1, \ldots, y_m) \leq 0$.

Prove that $Q(|y_1|, \ldots, |y_m|, 0, \ldots, 0) \leq 0$. □

Lemma

For each of the graphs on the following list, the corresponding quadratic form $Q(x_1, \ldots, x_\ell)$ *has determinant* 0*:*



Proof.

For the graphs $\Gamma = \tilde{A}_\ell$, each row of the symmetric matrix associated with the given quadratic form contains one entry 2 and two entries -1. The remaining entries are 0. Thus the sum of the columns is zero, and so $\det \tilde{A}_\ell = 0$.

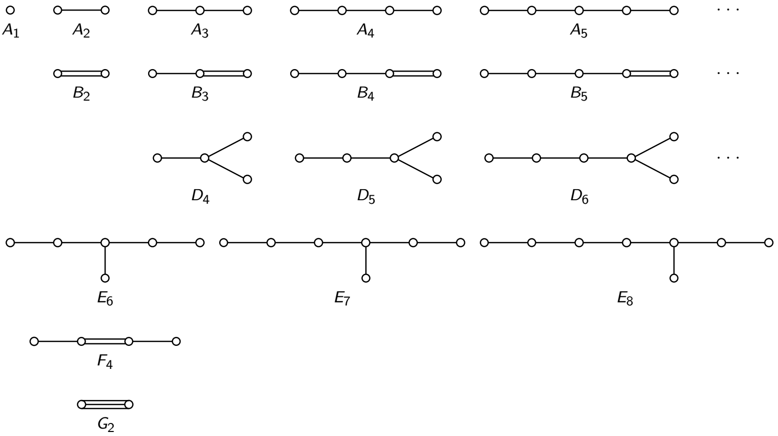
In all of the other graphs Γ on the list, we can find a vertex ℓ that is joined to just one vertex $\ell - 1$. We may choose ℓ such that it is connected to $\ell - 1$ by either a single or a double edge.

If there is a double edge, then we modify the way we obtained the previous formula to get the formula

$$\det \Gamma_\ell = 2 \det \Gamma_{\ell-1} - 2 \det \Gamma_{\ell-2}.$$

We calculate the determinants of the graphs inductively. □

Let Γ be a graph satisfying conditions (A), (B), (C). Then by the previous lemmas, Γ can have no subgraph of type $\tilde{A}_\ell$, $\tilde{B}_\ell$, $\tilde{C}_\ell$, $\tilde{D}_\ell$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$, $\tilde{F}_4$, or $\tilde{G}_2$. We use this information to show that Γ must be on the following list:



Corollary

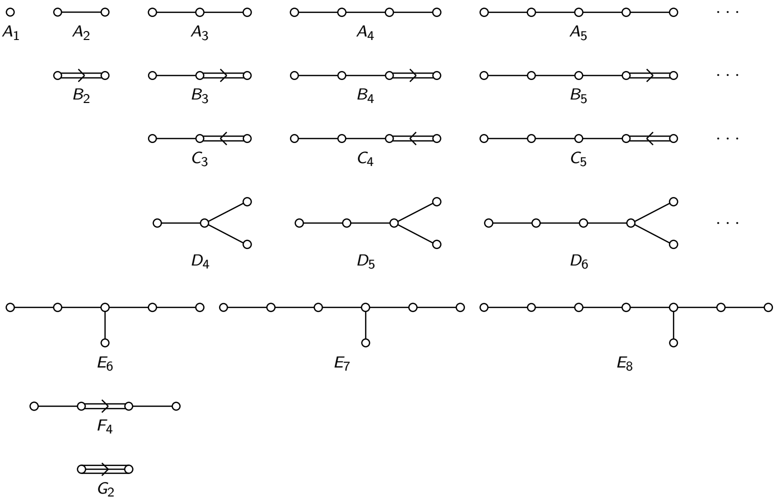
Let Δ *be the Dynkin diagram of a semisimple Lie algebra. Then each connected component of* Δ *must be one of the following graphs:*

$$A_\ell, \quad \ell \geq 1; \quad B_\ell, \quad \ell \geq 2; \quad D_\ell, \quad \ell \geq 4; \quad E_6; \quad E_7; \quad E_8; \quad F_4; \quad G_2.$$

We know that the Dynkin diagram is uniquely determined by the Cartan matrix. We can alter the Dynkin diagram so that it uniquely determines a Cartan matrix in the following way:

- We place an arrow on the double and triple edges.
- The arrow points from vertex i to vertex j if and only if $|\alpha_i| > |\alpha_j|$ in the Euclidean space V.
- This is equivalent to the condition $|A_{ji}| > |A_{ij}|$.

Thus the connected Dynkin diagrams are precisely the following:



Proposition

If a semisimple Lie algebra $\mathfrak{g}$ has a connected Dynkin diagram, then $\mathfrak{g}$ is simple.

Proof.

Let $\mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be a Cartan decomposition of $\mathfrak{g}$ giving rise to Δ. For each $\alpha \in \Phi$, choose a nonzero $e_\alpha \in \mathfrak{g}_\alpha$.

Let $\mathfrak{a}$ be a nonzero ideal of $\mathfrak{g}$. Show that $\mathfrak{h} \subset \mathfrak{a}$ using the fact that Δ is connected.

Then for each $\alpha \in \Phi$, we have

$$[h'_\alpha, e_\alpha] = \alpha\left(h'_\alpha\right) e_\alpha = \left\langle h'_\alpha, h'_\alpha \right\rangle e_\alpha.$$

Since $h'_\alpha \in \mathfrak{a}$, we have $[h'_\alpha, e_\alpha] \in \mathfrak{a}$.

Thus $e_\alpha \in \mathfrak{a}$ for all $\alpha \in \Phi$, and so $\mathfrak{g} \subset \mathfrak{a}$. □

Proposition

Let $\mathfrak{g}$ be a semisimple Lie algebra whose Dynkin diagram Δ splits into connected components $\Delta_1, \ldots, \Delta_r$. Then we have

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$$

where $\mathfrak{g}_i$ is a simple Lie algebra with Dynkin diagram Δ_i.

Proof.

We have $\Delta = \Delta_1 \dot\cup \Delta_2 \dot\cup \cdots \dot\cup \Delta_r$. Letting Π_i be the subset of Π corresponding to the vertices in Δ_i, we have

$$\Pi = \Pi_1 \dot\cup \Pi_2 \dot\cup \cdots \dot\cup \Pi_r.$$

Let $\mathfrak{h}_i$ be the subspace of $\mathfrak{h}$ spanned by the elements h'_α with $\alpha \in \Pi_i$. Then

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_r$$

where $\langle h, h' \rangle = 0$ if $h \in \mathfrak{h}_i$, $h' \in \mathfrak{h}_j$, and $i \neq j$.

For each i, let Φ_i be the set of all $\alpha \in \Phi$ such that $h'_\alpha \in \mathfrak{h}_i$. Then

$$\Phi = \Phi_1 \,\dot{\cup}\, \Phi_2 \,\dot{\cup}\, \cdots \,\dot{\cup}\, \Phi_r.$$

Define $\mathfrak{g}_i$ to be the subspace of $\mathfrak{g}$ spanned by $\mathfrak{h}_i$ and the e_α for all $\alpha \in \Phi_i$.

Given the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C} e_\alpha$ and the above decomposition of $\mathfrak{h}$, we see that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r,$$

a direct sum of subspaces.

Prove that each $\mathfrak{g}_i$ is a subalgebra of $\mathfrak{g}$.

Next, show that $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ if $i \neq j$.

Then each $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$ since

$$[\mathfrak{g}_i, \mathfrak{g}] = \sum_j [\mathfrak{g}_i, \mathfrak{g}_j] = [\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_i.$$

This implies that

$$[x_1 + \cdots + x_r, y_1 + \cdots + y_r] = [x_1, y_1] + \cdots + [x_r, y_r]$$

where $x_i, y_i \in \mathfrak{g}_i$. Thus

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$$

is a direct sum of Lie algebras.

If $\mathfrak{a}$ is a solvable ideal of $\mathfrak{g}_i$, then since since $[\mathfrak{a}, \mathfrak{g}_j] = 0$ for all $j \neq i$, we see that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$.

Because $\mathfrak{g}$ is semisimple, we have $\mathfrak{a} = 0$. Hence $\mathfrak{g}_i$ is semisimple.

Next, show that $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$.

Now consider the Cartan decomposition

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \sum_{\alpha \in \Phi_i} \mathbb{C} e_\alpha$$

of $\mathfrak{g}_i$ with respect to $\mathfrak{h}_i$.

Observe that Φ_i is the root system of $\mathfrak{g}_i$, that Π_i is a fundamental system in Φ_i, and that Δ_i is the Dynkin diagram of $\mathfrak{g}_i$.

Δ_i is connected, and so $\mathfrak{g}_i$ is simple by the previous proposition. $\square$

Corollary

A semisimple Lie algebra $\mathfrak{g}$ *has a connected Dynkin diagram if and only if* $\mathfrak{g}$ *is simple.*