The Cartan Decomposition of a Complex Semisimple Lie Algebra

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November 29, 2007

Definition

Let k be a field. A k-**algebra** is a k-vector space A equipped with a bilinear map $A \times A \to A$ called **multiplication**.

Definition

A **Lie algebra** is a k-algebra $\mathfrak{g}$ with multiplication $(x, y) \mapsto [x, y]$ satisfying the following conditions:

(i) $[x, x] = 0 \quad$ for all $x \in \mathfrak{g}$.

(ii) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad$ for all $x, y, z \in \mathfrak{g}$.

Condition (ii) is called the **Jacobi identity**.

Definition

We define the **lower central series** of $\mathfrak{g}$ recursively by

$$\mathfrak{g}^1 = \mathfrak{g}, \qquad \mathfrak{g}^{n+1} = [\mathfrak{g}^n, \mathfrak{g}] \quad \text{for } n \geq 1.$$

Proposition

Each $\mathfrak{g}^n$ is an ideal of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \mathfrak{g}^3 \supset \cdots$.

Definition

- We say that $\mathfrak{g}$ is **nilpotent** if $\mathfrak{g}^n = 0$ for some n.
- We say that $\mathfrak{g}$ is **abelian** if $\mathfrak{g}^2 = 0$.

Definition

We define the **derived series** of $\mathfrak{g}$ recursively by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}^{(n+1)} = \left[\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}\right] \quad \text{for } n \geq 0.$$

Proposition

Each $\mathfrak{g}^{(n)}$ is an ideal of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{g}^{(0)} \supset \mathfrak{g}^{(1)} \supset \mathfrak{g}^{(2)} \supset \cdots$.

Definition

A Lie algebra $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(n)} = 0$ for some n.

Proposition

Every nilpotent Lie algebra is solvable.

Proposition

Every finite-dimensional Lie algebra contains a unique maximal solvable ideal $\mathfrak{r}$. (This is called the **solvable radical**.)

Definition

A Lie algebra $\mathfrak{g}$ is **semisimple** if $\mathfrak{r} = 0$.

Definition

A Lie algebra $\mathfrak{g}$ is **simple** if it contains no ideals other than itself and the zero ideal.

The 1-dimensional Lie algebra is called the **trivial simple Lie algebra**.

Proposition

Every non-trivial simple Lie algebra is semisimple.

Let $M_m(k)$ denote the ring of all $n \times n$ matrices over k. We define $\mathfrak{gl}_n(k)$ to be the Lie algebra $[M_n(k)]$ formed from $M_n(k)$ via the commutator product. We denote this Lie algebra by $\mathfrak{gl}_n(k)$.

Definition

A **representation** of a Lie algebra $\mathfrak{g}$ is a homomorphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}_n(k)$.

Definition

A $\mathfrak{g}$-module is a k-vector space V equipped with a left $\mathfrak{g}$-action $\mathfrak{g} \times V \to V$ satisfying the following properties:

- $(x, v) \mapsto xv$ is linear in x and v.
- $[x, y]v = x(yv) - y(xv) \quad$ for all $x, y \in \mathfrak{g}$, $v \in V$.

Example

The vector space $\mathfrak{g}$ forms a $\mathfrak{g}$-module via the action $(x, y) \mapsto [x, y]$.

- We have $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$ by the Jacobi identity.
- We call this module the **adjoint module** and denote the action of x on the vector y by $\operatorname{ad} x \cdot y$.
- We have $\operatorname{ad}[x, y] = \operatorname{ad} x \operatorname{ad} y - \operatorname{ad} y \operatorname{ad} x$.

From now on, we assume that $\mathfrak{g}$ is a finite-dimensional Lie algebra over the field $\mathbb{C}$ of complex numbers.

Theorem (Lie's theorem)

Let $\mathfrak{g}$ *be a solvable Lie algebra and* V *a finite-dimensional irreducible* $\mathfrak{g}$*-module. Then* $\dim V = 1$*.*

Corollary

Let $\mathfrak{g}$ *be a solvable Lie algebra and* V *a finite-dimensional* $\mathfrak{g}$*-module. Then a basis can be chosen for* V *with respect to which we obtain a matrix representation* ρ *of* $\mathfrak{g}$ *of the form*

$$\rho(x) = \begin{pmatrix} * & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & * \end{pmatrix} \qquad \text{for all } x \in \mathfrak{g}.$$

Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation with eigenvalues $\lambda_1, \ldots, \lambda_r$.

The **generalized eigenspace** of V with respect to λ_i is the set of all $v \in V$ annihilated by some power of $T - \lambda_i 1$.

We have

- $V = V_1 \oplus \cdots \oplus V_r$.
- Each V_i is invariant under the action of T.

Theorem

Let $\mathfrak{g}$ be a nilpotent Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Then for any $y \in \mathfrak{g}$, the generalized eigenspaces of V associated with $\rho(y)$ are all submodules of V.

Corollary

Let $\mathfrak{g}$ be a nilpotent Lie algebra and V a finite-dimensional indecomposable $\mathfrak{g}$-module. Then a basis can be chosen for V with respect to which we obtain a representation ρ of $\mathfrak{g}$ of the form

$$\rho(x) = \begin{pmatrix} \lambda(x) & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ 0 & & & \lambda(x) \end{pmatrix} \qquad \text{for all } x \in \mathfrak{g}.$$

Notice that $\lambda : x \mapsto \lambda(x)$ is a 1-dimensional representation of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. For any 1-dimensional representation λ of $\mathfrak{g}$, we define the set

$$V_\lambda = \{v \in V \mid (\forall x \in \mathfrak{g})(\exists N(x) \geq 1) \quad (\rho(x) - \lambda(x)1)^{N(x)} v = 0\}.$$

Theorem

$V = \bigoplus_\lambda V_\lambda$, and each V_λ is a submodule of V.

Definition

If $V_\lambda \neq 0$, then we call λ a **weight** of $\mathfrak{g}$ and V_λ the **weight space** of λ.

We call $V = \bigoplus_\lambda V_\lambda$ the **weight space decomposition** of V.

Since each V_λ is the direct sum of the indecomposable components giving rise to λ, it follows that a basis can be chosen for V_λ with respect to which a representation ρ of $\mathfrak{g}$ on V_λ has the form

$$\rho(x) = \begin{pmatrix} \lambda(x) & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ 0 & & & \lambda(x) \end{pmatrix} \qquad \text{for all } x \in \mathfrak{g}.$$

Theorem (Engel's theorem)

A Lie algebra $\mathfrak{g}$ *is nilpotent if and only if* $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ *is nilpotent for all* $x \in \mathfrak{g}$.

Corollary

A Lie algebra $\mathfrak{g}$ *is nilpotent if and only if* $\mathfrak{g}$ *has a basis with respect to which the adjoint representation of* $\mathfrak{g}$ *has the form*

$$\rho(x) = \begin{pmatrix} 0 & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 0 \end{pmatrix} \quad \text{for all } x \in \mathfrak{g}.$$

Notice that if $\mathfrak{h}$ is a subalgebra of a Lie algebra $\mathfrak{g}$, then $\mathfrak{h}$ induces an $\mathfrak{h}$-module structure on $\mathfrak{g}$ via the adjoint action.

If $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is chosen carefully, then $\mathfrak{h}$ induces weight space decomposition of $\mathfrak{g}$ that tells us a lot about a Lie algebra's structure.

Definition

Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. We define the **normalizer** of $\mathfrak{h}$ to be the set

$$N(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [h, x] \in \mathfrak{h} \quad \text{for all } h \in \mathfrak{h}\}.$$

Proposition

$N(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{h}$ is an ideal of $N(\mathfrak{h})$, and $N(\mathfrak{h})$ is the largest subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ as an ideal.

Proof.

- If $x, y \in N(\mathfrak{h})$, $h \in \mathfrak{h}$, then

$$[h, [x, y]] = [[y, h], x] + [[h, x], y] \in \mathfrak{h}$$

  by the Jacobi identity, and so $N(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$.
- $\mathfrak{h}$ is clearly an ideal of $N(\mathfrak{h})$.
- If $\mathfrak{h}$ is an ideal of $\mathfrak{m}$, then $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{h}$ so that $\mathfrak{m} \subset N(\mathfrak{h})$. $\square$

Definition

A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a **Cartan subalgebra** of $\mathfrak{g}$ if $\mathfrak{h}$ is nilpotent and $N(\mathfrak{h}) = \mathfrak{h}$.

Definition

Let $x \in \mathfrak{g}$. The **null component** $\mathfrak{g}_{0,x}$ of $\mathfrak{g}$ with respect to x is the generalized eigenspace of $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$, that is,

$$\mathfrak{g}_{0,x} = \{y \in \mathfrak{g} \mid (\operatorname{ad} x)^n y = 0 \quad \text{for some } n \geq 1\}.$$

Definition

An element $x \in \mathfrak{g}$ is **regular** if $\dim \mathfrak{g}_{0,x}$ is as small as possible.

Any Lie algebra will certainly contain regular elements.

Theorem

If x is a regular element of $\mathfrak{g}$, then $\mathfrak{g}_{0,x}$ is a Cartan subalgebra of $\mathfrak{g}$.

Definition

A **derivation** of a Lie algebra $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$D[x, y] = [Dx, y] + [x, Dy] \qquad \text{for all } x, y \in \mathfrak{g}.$$

Proposition

ad x *is a derivation for all* $x \in \mathfrak{g}$.

Proof.

$\mathrm{ad} x \cdot [y, z] = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [\mathrm{ad} x \cdot y, z] + [y, \mathrm{ad} x \cdot z]$. □

The automorphisms of $\mathfrak{g}$ form a group $\mathrm{Aut}(\mathfrak{g})$ under composition of maps.

Proposition

If D is a nilpotent derivation of $\mathfrak{g}$, then $\exp(D)$ is an automorphism of $\mathfrak{g}$.

Definition

An **inner automorphism** of $\mathfrak{g}$ is an automorphism of the form $\exp(\operatorname{ad} x)$ for $x \in \mathfrak{g}$ with $\operatorname{ad} x$ nilpotent. The **inner automorphism group** is the subgroup $\mathrm{Inn}(\mathfrak{g})$ of $\mathrm{Aut}(\mathfrak{g})$ generated by all inner automorphisms.

Proposition

$\mathrm{Inn}(\mathfrak{g})$ *is a normal subgroup of* $\mathrm{Aut}(\mathfrak{g})$.

Definition

Two subalgebras $\mathfrak{h}, \mathfrak{k}$ are **conjugate** in $\mathfrak{g}$ if there exists a $\phi \in \mathrm{Inn}(\mathfrak{g})$ such that $\phi(\mathfrak{h}) = \mathfrak{k}$.

Theorem

Any two Cartan subalgebras of $\mathfrak{g}$ are conjugate.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Since $\mathfrak{h}$ is nilpotent, the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ induces a weight space decomposition $\mathfrak{g} = \bigoplus_\lambda \mathfrak{g}_\lambda$ where

$$\mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid (\forall h \in \mathfrak{h})(\exists n \geq 1) \quad (\mathrm{ad}\, h - \lambda(h)1)^n x = 0\}.$$

Proposition

$\mathfrak{h} = \mathfrak{g}_0$.

Proof.

Since $\mathfrak{h}$ is nilpotent, we can choose a basis of $\mathfrak{g}$ with respect to which $\operatorname{ad} x$ is represented by a strict upper-triangular matrix for all $x \in \mathfrak{h}$. This follows from the corollary to Engel's theorem.

Each such matrix has eigenvalue zero, and so $\mathfrak{h} \subset \mathfrak{g}_0$.

Now suppose $\mathfrak{h} \neq \mathfrak{g}_0$ and let $\mathfrak{m}/\mathfrak{h}$ be an irreducible $\mathfrak{h}$-submodule of $\mathfrak{g}_0/\mathfrak{h}$.

By Lie's theorem, we have $\dim \mathfrak{m}/\mathfrak{h} = 1$. The 1-dimensional representation induced by $\mathfrak{m}/\mathfrak{h}$ must be the zero map since $\mathfrak{h}$ is nilpotent.

Hence $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{h}$, and so $\mathfrak{m} \subset N(\mathfrak{h})$. This contradicts the fact that $\mathfrak{h} = N(\mathfrak{h})$. □

Thus we obtain a decomposition of $\mathfrak{g}$ of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_{\lambda_1} \oplus \cdots \oplus \mathfrak{g}_{\lambda_r} \qquad \lambda_1, \ldots, \lambda_r \neq 0.$$

Definition

A 1-dimensional representation λ of $\mathfrak{h}$ is called a **root** of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. We denote the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$ by Φ. Thus

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

We call this decomposition the **Cartan decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Each $\mathfrak{g}_\alpha$ is called the **root space** of α.

Proposition

If λ and μ are 1-dimensional representations of $\mathfrak{h}$, then $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

Proof.

Let $y \in \mathfrak{g}_\lambda$ and $z \in \mathfrak{g}_\mu$. If $x \in \mathfrak{h}$, then

$$(\mathrm{ad}x - \lambda(x)1 - \mu(x)1)^n[y,z] = \sum_{i=0}^{n} \binom{n}{i} \left[(\mathrm{ad}\,x - \lambda(x)1)^i y, (\mathrm{ad}\,x - \mu(x)1)^{n-i} z\right].$$

Hence $(\mathrm{ad}\,x - \lambda(x)1 - \mu(x)1)^n[y,z] = 0$ if n is sufficiently large. □

Corollary

If $\alpha, \beta \in \Phi$ are roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, then

$$\begin{aligned} [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &\subset \mathfrak{g}_{\alpha+\beta} && \text{if } \alpha+\beta \in \Phi \\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &\subset \mathfrak{h} && \text{if } \beta = -\alpha \\ [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] &= 0 && \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Phi. \end{aligned}$$

Proposition

Let $\alpha \in \Phi$. Given any $\beta \in \Phi$, there exists a number $r \in \mathbb{Q}$, depending on α and β, such that $\beta = r\alpha$ on the subspace $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$ of $\mathfrak{h}$.

Proof.

If $-\alpha$ is not a weight of $\mathfrak{g}$ with respect to $\mathfrak{h}$, then $\mathfrak{g}_{-\alpha} = 0$, and the proof is trivial.

So assume $-\alpha$ is a weight. Then since $\alpha \neq 0$, we must have $-\alpha \in \Phi$. For $i \in \mathbb{Z}$, we consider the function $i\alpha + \beta : \mathfrak{h} \to \mathbb{C}$. Since Φ is finite, there exist integers p and q with $p \geq 0$ and $q \geq 0$ such that

$$-p\alpha + \beta, \ldots, \beta, \ldots, q\alpha + \beta$$

are all in Φ but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ are not in Φ.

If either $-(p+1)\alpha+\beta=0$ or $(q+1)\alpha+\beta=0$, then the result is obvious.

So assume $-(p+1)\alpha+\beta\neq 0$ and $(q+1)\alpha+\beta\neq 0$. Let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ given by

$$\mathfrak{m}=\mathfrak{g}_{-p\alpha+\beta}\oplus\cdots\oplus\mathfrak{g}_{q\alpha+\beta}.$$

Let $x=[y,z]$ with $y\in\mathfrak{g}_\alpha$ and $z\in\mathfrak{g}_{-\alpha}$. We have $\operatorname{ad} y\cdot\mathfrak{g}_{q\alpha+\beta}\subset\mathfrak{g}_{(q+1)\alpha+\beta}$. Because $(q+1)\alpha+\beta\neq 0$ and $(q+1)\alpha+\beta\notin\Phi$, we must have $\mathfrak{g}_{(q+1)\alpha+\beta}=0$.

Thus $\operatorname{ad} y\cdot\mathfrak{m}\subset\mathfrak{m}$. By a similar argument, we have $\operatorname{ad} z\cdot\mathfrak{m}\subset\mathfrak{m}$, and so

$$\operatorname{ad} x\cdot\mathfrak{m}=(\operatorname{ad} y\ \operatorname{ad} z-\operatorname{ad} z\ \operatorname{ad} y)\mathfrak{m}\subset\mathfrak{m}.$$

We calculate the trace $\mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\, x)$. Since $x \in \mathfrak{h}$, each weight space $\mathfrak{g}_{i\alpha+\beta}$ is invariant under $\mathrm{ad}\, x$. Thus

$$\mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\, x) = \sum_{i=-p}^{q} \mathrm{tr}_{\mathfrak{g}_{i\alpha+\beta}}(\mathrm{ad}\, x).$$

Now $\mathrm{ad}\, x$ acts on $\mathfrak{g}_{i\alpha+\beta}$ via a matrix of the form

$$\begin{pmatrix} (i\alpha+\beta)(x) & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & (i\alpha+\beta)(x) \end{pmatrix}.$$

Thus $\mathrm{tr}_{\mathfrak{g}_{i\alpha+\beta}}(\mathrm{ad}\, x) = \dim \mathfrak{g}_{i\alpha+\beta}(i\alpha+\beta)(x)$.

It follows that

$$\operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} x) = \sum_{i=-p}^{q} \dim \mathfrak{g}_{i\alpha+\beta}(i\alpha+\beta)(x)$$

$$= \left(\sum_{i=-p}^{q} i \dim \mathfrak{g}_{i\alpha+\beta}\right)\alpha(x) + \left(\sum_{i=-p}^{q} \dim \mathfrak{g}_{i\alpha+\beta}\right)\beta(x).$$

But we also have

$$\operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} x) = \operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} y \operatorname{ad} z - \operatorname{ad} z \operatorname{ad} y) = \operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} y \operatorname{ad} z) - \operatorname{tr}_{\mathfrak{m}}(\operatorname{ad} z \operatorname{ad} y) = 0.$$

Hence

$$\left(\sum_{i=-p}^{q} i \dim \mathfrak{g}_{i\alpha+\beta}\right)\alpha(x) + \left(\sum_{i=-p}^{q} \dim \mathfrak{g}_{i\alpha+\beta}\right)\beta(x) = 0.$$

We know that $\dim \mathfrak{g}_{i\alpha+\beta} > 0$ for all $-p \leq i \leq q$. Thus

$$\beta(x) = \frac{\left(\sum_{i=-p}^{q} i \dim \mathfrak{g}_{i\alpha+\beta}\right)}{\left(\sum_{i=-p}^{q} \dim \mathfrak{g}_{i\alpha+\beta}\right)}\alpha(x).$$

□

Definition

We define the **Killing form** of $\mathfrak{g}$ to be the bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $\langle x, y \rangle = \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$.

Proposition

(i) *The Killing form is symmetric, i.e.,* $\langle x, y \rangle = \langle y, x \rangle$ *for all* $x, y \in \mathfrak{g}$*.*

(ii) *The Killing form is invariant, i.e.,* $\langle [x, y], z \rangle = \langle x, [y, z] \rangle$ *for all* $x, y, z \in \mathfrak{g}$*.*

Proposition

Let $\mathfrak{a}$ *be an ideal of* $\mathfrak{g}$ *and let* $x, y \in \mathfrak{a}$*. Then* $\langle x, y \rangle_{\mathfrak{a}} = \langle x, y \rangle_{\mathfrak{g}}$*. Hence the killing form of* $\mathfrak{g}$ *restricted to* $\mathfrak{a}$ *is the Killing form of* $\mathfrak{a}$*.*

Proof.

We choose a basis of $\mathfrak{a}$ and extend it to a basis of $\mathfrak{g}$. With respect to this basis, $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ is represented by a matrix of the form

$$\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$$

since $x \in \mathfrak{a}$.

Similarly, $\operatorname{ad} y : \mathfrak{g} \to \mathfrak{g}$ is represented by a matrix of the form

$$\begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}.$$

Thus $\mathrm{ad}\, x\ \mathrm{ad}\, y : \mathfrak{g} \to \mathfrak{g}$ is represented by the matrix

$$\begin{pmatrix} A_1 B_1 & A_1 B_2 \\ 0 & 0 \end{pmatrix}.$$

Hence $\mathrm{tr}_{\mathfrak{a}}(\mathrm{ad}\, x\ \mathrm{ad}\, y) = \mathrm{tr}(A_1 B_1) = \mathrm{tr}_{\mathfrak{g}}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$, and so

$\langle x, y \rangle_{\mathfrak{a}} = \langle x, y \rangle_{\mathfrak{g}}$. □

Proposition

If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$.

Proof.

If $[x, y] \in [\mathfrak{a}^{\perp}, \mathfrak{g}]$ with $x \in \mathfrak{a}^{\perp}$ and $y \in \mathfrak{g}$, then for all $z \in \mathfrak{a}$, we have

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle = 0.$$ □

In particular, $\mathfrak{g}^{\perp}$ is an ideal of $\mathfrak{g}$.

Definition

The Killing form of $\mathfrak{g}$ is **nondegenerate** if $\mathfrak{g}^\perp = 0$. The Killing form of $\mathfrak{g}$ is **identically zero** if $\mathfrak{g}^\perp = \mathfrak{g}$.

Proposition

Let $\mathfrak{g}$ be a Lie algebra such that $\mathfrak{g} \neq 0$ and $\mathfrak{g}^2 = \mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then there exists an $x \in \mathfrak{h}$ such that $\langle x, x \rangle \neq 0$.

Proof.

Let $\mathfrak{g} = \oplus \mathfrak{g}_\lambda$ be the Cartan decomposition of $\mathfrak{g}$. Then

$$\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] = \left[\bigoplus_\lambda \mathfrak{g}_\lambda, \bigoplus_\lambda \mathfrak{g}_\lambda \right] = \sum_{\lambda,\mu} [\mathfrak{g}_\lambda, \mathfrak{g}_\mu].$$

We have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$. Thus $[\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{h}$, while $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu]$ is contained in the complement of $\mathfrak{h}$ in $\mathfrak{g}$ if $\mu \neq -\lambda$.

Since $\mathfrak{g} = \mathfrak{g}^2$, we must have

$$\mathfrak{h} = \sum_{\lambda} [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}]$$

summed over all weights λ such that $-\lambda$ is also a weight.

Thus

$$\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] + \sum_{\alpha} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$$

summed over all roots α such that $-\alpha$ is also a root.

Note that $\mathfrak{g}$ is not nilpotent since $\mathfrak{g}^2 = \mathfrak{g} \neq 0$. But we know that $\mathfrak{h}$ is nilpotent, and so $\mathfrak{h} \neq \mathfrak{g}$. Thus there exists at least one root $\beta \in \Phi$.

Now β is a 1-dimensional representation of $\mathfrak{h}$, and so β vanishes on $[\mathfrak{h}, \mathfrak{h}]$. But β does not vanish on $\mathfrak{h}$ since $\beta \neq 0$.

Using the above decomposition of $\mathfrak{h}$, we see there exists some root $\alpha \in \Phi$ such that $-\alpha \in \Phi$ and β does not vanish on $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Choose an $x \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ such that $\beta(x) \neq 0$. Then

$$\langle x, x \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} x) = \sum_\lambda \dim \mathfrak{g}_\lambda (\lambda(x))^2$$

since $\operatorname{ad} x$ is represented on $\mathfrak{g}_\lambda$ by a matrix of the form

$$\begin{pmatrix} \lambda(x) & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \lambda(x) \end{pmatrix}.$$

For each λ, there exists an $r_{\lambda,\alpha} \in \mathbb{Q}$ such that $\lambda(x) = r_{\lambda,\alpha}\alpha(x)$.

Thus

$$\langle x, x \rangle = \left(\sum_{\lambda} \dim \mathfrak{g}_\lambda r_{\lambda,\alpha}^2 \right) \alpha(x)^2.$$

Now $\beta(x) = r_{\beta,\alpha}\alpha(x)$ and $\beta(x) \neq 0$. Thus $r_{\beta,\alpha} \neq 0$ and $\alpha(x) \neq 0$. It follows that $\langle x, x \rangle \neq 0$. □

Theorem

If the Killing form of $\mathfrak{g}$ is identically zero, then $\mathfrak{g}$ is solvable.

Proof.

We proceed by induction on $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$, then $\mathfrak{g}$ is clearly solvable. So assume $\dim \mathfrak{g} > 1$.

By the contrapositive of the last proposition, we see that $\mathfrak{g} \neq \mathfrak{g}^2$. Now $\mathfrak{g}^2$ is an ideal of $\mathfrak{g}$, so the Killing form of $\mathfrak{g}^2$ is the restriction of the Killing form of $\mathfrak{g}$.

Hence the Killing form of $\mathfrak{g}^2$ is identically zero. It follows by induction that $\mathfrak{g}^2$ is solvable. We also have $\left(\mathfrak{g}/\mathfrak{g}^2\right)^2 = 0$, and so $\mathfrak{g}/\mathfrak{g}^2$ is solvable. Thus $\mathfrak{g}$ is solvable. □

Theorem (Cartan's criterion)

A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate.

Proof.

We prove the contrapositive. If the Killing form of $\mathfrak{g}$ is degenerate, then $\mathfrak{g}^\perp \neq 0$.

We know that $\mathfrak{g}^\perp$ is an ideal, and thus the Killing form of $\mathfrak{g}^\perp$ is identically zero. This implies $\mathfrak{g}^\perp$ is solvable by the last theorem.

Thus $\mathfrak{g}$ has a nonzero solvable ideal, and so $\mathfrak{g}$ is not semisimple.

Now suppose $\mathfrak{g}$ is not semisimple. Then the solvable radical $\mathfrak{r}$ of $\mathfrak{g}$ is nonzero. Consider the chain of subspaces

$$\mathfrak{r} = \mathfrak{r}^{(0)} \supset \mathfrak{r}^{(1)} \supset \mathfrak{r}^{(2)} \supset \cdots \supset \mathfrak{r}^{(k-1)} \supset \mathfrak{r}^{(k)} = 0.$$

Each subspace $\mathfrak{r}^{(i)}$ is an ideal of $\mathfrak{g}$ since the product of two ideals is an ideal.

Let $\mathfrak{a} = \mathfrak{r}^{(k-1)}$. Then $\mathfrak{a}$ is a nonzero ideal such that $\mathfrak{a}^2 = 0$. We choose a basis of $\mathfrak{a}$ and extend it to a basis of $\mathfrak{g}$.

Let $x \in \mathfrak{a}$, $y \in \mathfrak{g}$. With respect to our chosen basis, $\operatorname{ad} x$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

since $\mathfrak{a}^2 = 0$ and $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, and $\operatorname{ad} y$ is represented by a matrix of the form

$$\begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}.$$

Thus $\operatorname{ad} x \operatorname{ad} y$ is represented by the matrix

$$\begin{pmatrix} 0 & AB_3 \\ 0 & 0 \end{pmatrix}.$$

Hence $\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y) = 0$. This holds for all $x \in \mathfrak{a}$, $y \in \mathfrak{g}$, and so $\mathfrak{a} \subset \mathfrak{g}^{\perp}$. Thus $\mathfrak{g}^{\perp} \neq 0$, and so the Killing form of $\mathfrak{g}$ is degenerate. □

From now on, we assume that $\mathfrak{g}$ is a semisimple Lie algebra.

Proposition

If $\mu \neq -\lambda$, then $\mathfrak{g}_\lambda$ and $\mathfrak{g}_\mu$ are orthogonal with respect to the Killing form.

Proof.

Let $x \in \mathfrak{g}_\lambda$, $y \in \mathfrak{g}_\mu$. For every weight space $\mathfrak{g}_\nu$, we have

$$\mathrm{ad}\, x\ \mathrm{ad}\, y \cdot \mathfrak{g}_\nu \subset \mathfrak{g}_{\lambda+\mu+\nu}.$$

We choose a basis of $\mathfrak{g}$ adapted to the Cartan decomposition. With respect to such a basis, $\operatorname{ad} x \operatorname{ad} y$ is represented by a block matrix of the form

$$\begin{pmatrix} 0 & & & & * \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ * & & & & 0 \end{pmatrix}$$

since $\lambda + \mu + \nu \neq \nu$.

It follows that $\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = 0$, and so $\mathfrak{g}_\lambda$ is orthogonal to $\mathfrak{g}_\mu$.

□

Proposition

If α is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, then $-\alpha$ is also a root.

Proof.

Recall that α is a root if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq 0$. Suppose $-\alpha$ is not a root.

Then since $-\alpha \neq 0$, we must have $\mathfrak{g}_{-\alpha} = 0$. This implies that $\mathfrak{g}_\alpha$ is orthogonal to all $\mathfrak{g}_\lambda$, and thus $\mathfrak{g}_\alpha \subset \mathfrak{g}^\perp$.

But $\mathfrak{g}$ is semisimple, and so $\mathfrak{g}^\perp = 0$ by Cartan's criterion. Thus $\mathfrak{g}_\alpha = 0$, contradicting the fact that α is a root. □

Proposition

The Killing form of $\mathfrak{g}$ remains nondegenerate on restriction to $\mathfrak{h}$.

Proof.

Let $x \in \mathfrak{h}$ and suppose $\langle x, y \rangle = 0$ for all $y \in \mathfrak{h}$. We also have $\langle x, y \rangle = 0$ for all $y \in \mathfrak{g}_\alpha$ where $\alpha \in \Phi$.

Thus $\langle x, y \rangle = 0$ for all $y \in \mathfrak{g}$, and so $x \in \mathfrak{g}^\perp$. But $\mathfrak{g}^\perp = 0$ since $\mathfrak{g}$ is semisimple, and so $x = 0$. □

Theorem

The Cartan subalgebras of a semisimple Lie algebra are abelian.

Proof.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For all $x \in [\mathfrak{h}, \mathfrak{h}]$, $y \in \mathfrak{h}$, we have

$$\langle x, y \rangle = \operatorname{tr}(\operatorname{ad} x \operatorname{ad} y) = \sum_{\lambda} \dim \mathfrak{g}_{\lambda} \lambda(x)\lambda(y)$$

since $\operatorname{ad} x \operatorname{ad} y$ is represented on $\mathfrak{g}_{\lambda}$ by a matrix of the form

$$\begin{pmatrix} \lambda(x)\lambda(y) & & & * \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot & \\ 0 & & & \lambda(x)\lambda(y) \end{pmatrix}.$$

But λ is a 1-dimensional representation of $\mathfrak{h}$, and so λ vanishes on $[\mathfrak{h}, \mathfrak{h}]$.

Thus $\lambda(x) = 0$.

It follows that $\langle x, y \rangle = 0$ for all $y \in \mathfrak{h}$.

Since the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate, this implies $x = 0$. Hence $[\mathfrak{h}, \mathfrak{h}] = 0$, and so $\mathfrak{h}$ is abelian. □

Let $\mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ be the dual space of $\mathfrak{h}$. We have $\dim \mathfrak{h}^* = \dim \mathfrak{h}$.

We define a map $\mathfrak{h} \to \mathfrak{h}^*$ using the Killing form of $\mathfrak{g}$. Given $h \in \mathfrak{h}$, we define $h^* \in \mathfrak{h}^*$ by

$$h^*(x) = \langle h, x \rangle \qquad \text{for all } x \in \mathfrak{h}.$$

Lemma

The map $h \mapsto h^*$ is an isomorphism of vector spaces between $\mathfrak{h}$ and $\mathfrak{h}^*$.

Notice that Φ is a finite subset of $\mathfrak{h}^*$.

Because the map $h \mapsto h^*$ is bijective, we know that for each $\alpha \in \Phi$, there exists a unique element $h'_\alpha \in \mathfrak{h}$ such that $h'^*_\alpha(x) = \alpha(x)$ for all $x \in \mathfrak{h}$, that is,

$$\alpha(x) = \left\langle h'_\alpha, x \right\rangle \qquad \text{for all } x \in \mathfrak{h}.$$

Proposition

The vectors h'_α for $\alpha \in \Phi$ span $\mathfrak{h}$.

Proof.

Suppose the vectors h'_α are contained in a proper subspace of $\mathfrak{h}$. Then the annihilator of this subspace is nonzero.

Thus there exists a nonzero $x \in \mathfrak{h}$ such that $x^* \left(h'_\alpha\right) = 0$ for all $\alpha \in \Phi$, that is, $\left\langle h'_\alpha, x \right\rangle = 0$. Hence $\alpha(x) = 0$ for all $\alpha \in \Phi$.

Let $y \in \mathfrak{h}$. Then

$$\langle x, y \rangle = \mathrm{tr}(\mathrm{ad}\, x \ \mathrm{ad}\, y) = \sum_{\lambda} \dim \mathfrak{g}_{\lambda} \ \lambda(x)\lambda(y) = 0$$

since $\lambda(x) = 0$ for all weights λ.

Thus $\langle x, y \rangle = 0$ for all $y \in \mathfrak{h}$. Since the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate, this implies $x = 0$, a contradiction. □

Proposition

$h'_{\alpha} \in [\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$ *for all* $\alpha \in \Phi$.

Proof.

We know that $\mathfrak{g}_{\alpha}$ is an $\mathfrak{h}$-module. Since all irreducible $\mathfrak{h}$-modules are 1-dimensional, $\mathfrak{g}_{\alpha}$ contains a 1-dimensional $\mathfrak{h}$-submodule $\mathbb{C}e_{\alpha}$.

We have $[x, e_{\alpha}] = \alpha(x)e_{\alpha}$ for all $x \in \mathfrak{h}$.

Let $y \in \mathfrak{g}_{-\alpha}$. Then $[e_\alpha, y] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$. I claim that $[e_\alpha, y] = \langle e_\alpha, y \rangle h'_\alpha$.

We define the element

$$z = [e_\alpha, y] - \langle e_\alpha, y \rangle h'_\alpha \in \mathfrak{h}.$$

Let $x \in \mathfrak{h}$. Then

$$\begin{aligned} \langle x, z \rangle &= \langle x, [e_\alpha, y] \rangle - \langle e_\alpha, y \rangle \langle x, h'_\alpha \rangle \\ &= \langle [x, e_\alpha], y \rangle - \langle e_\alpha, y \rangle \alpha(x) \\ &= \alpha(x) \langle e_\alpha, y \rangle - \langle e_\alpha, y \rangle \alpha(x) = 0. \end{aligned}$$

Thus $\langle x, z \rangle = 0$ for all $x \in \mathfrak{h}$. Since the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate, this implies $z = 0$. Hence $[e_\alpha, y] = \langle e_\alpha, y \rangle h'_\alpha$ for all $y \in \mathfrak{g}_{-\alpha}$.

Now there exists a $y \in \mathfrak{g}_{-\alpha}$ such that $\langle e_\alpha, y \rangle \neq 0$. For otherwise e_α would be orthogonal to $\mathfrak{g}_{-\alpha}$, and thus to the whole of $\mathfrak{g}$.

This would imply $e_\alpha \in \mathfrak{g}^\perp$. But $\mathfrak{g}^\perp = 0$ since $\mathfrak{g}$ is semisimple, and so $e_\alpha = 0$, a contradiction.

Choosing a $y \in \mathfrak{g}_{-\alpha}$ such that $\langle e_\alpha, y \rangle \neq 0$, we have

$$h'_\alpha = \frac{1}{\langle e_\alpha, y \rangle} [e_\alpha, y] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}].$$

□

Proposition

$\langle h'_\alpha, h'_\alpha \rangle \neq 0$ for all $\alpha \in \Phi$.

Proof.

Suppose $\langle h'_\alpha, h'_\alpha \rangle = 0$ for some $\alpha \in \Phi$. Let β be any element of Φ.

There exists an $r_{\beta,\alpha} \in \mathbb{Q}$ such that $\beta = r_{\beta,\alpha}\alpha$ on $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Now $h'_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Thus

$$\beta(h'_\alpha) = r_{\beta,\alpha}\alpha(h'_\alpha),$$

that is, $\langle h'_\beta, h'_\alpha \rangle = r_{\beta,\alpha}\langle h'_\alpha, h'_\alpha \rangle = 0$.

This holds for all $\beta \in \Phi$. But the vectors h'_α for $\alpha \in \Phi$ span $\mathfrak{h}$, and so $\langle x, h'_\alpha \rangle = 0$ for all $x \in \mathfrak{h}$.

Since the Killing form of $\mathfrak{g}$ restricted to $\mathfrak{h}$ is nondegenerate, this implies $h'_\alpha = 0$. Thus $\alpha = 0$, contradicting the fact that $\alpha \in \Phi$. □

Theorem

$\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.

Proof.

Choose a 1-dimensional $\mathfrak{h}$-submodule $\mathbb{C}e_\alpha$ of $\mathfrak{g}_\alpha$. We can find an $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that $[e_\alpha, e_{-\alpha}] = h'_\alpha$.

Consider the subspace $\mathfrak{m}$ of $\mathfrak{g}$ given by

$$\mathfrak{m} = \mathbb{C}e_\alpha \oplus \mathbb{C}h'_\alpha \oplus \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_{-2\alpha} \oplus \cdots .$$

There are only finitely-many summands of $\mathfrak{m}$ since Φ is finite. Thus there are only finitely-many non-negative integers r such that $\mathfrak{g}_{-r\alpha} = 0$.

Observe that $\text{ad}\, e_\alpha \cdot \mathfrak{m} \subset \mathfrak{m}$ because

$$[e_\alpha, e_\alpha] = 0,$$

$$[e_\alpha, h'_\alpha] = -\alpha\,(h'_\alpha)\, e_\alpha,$$

$$[e_\alpha, y] = \langle e_\alpha, y\rangle h'_\alpha \qquad \text{for all } y \in \mathfrak{g}_{-\alpha}.$$

and

$$\text{ad}\, e_\alpha \cdot \mathfrak{g}_{-r\alpha} \subset \mathfrak{g}_{-(r-1)\alpha} \qquad \text{for all } r \geq 2.$$

Similarly, $\text{ad}\, e_{-\alpha} \cdot \mathfrak{m} \subset \mathfrak{m}$ because

$$[e_{-\alpha}, e_\alpha] = h'_\alpha,$$

$$[e_{-\alpha}, h'_\alpha] = \alpha\,(h'_\alpha)\, e_{-\alpha},$$

and $\text{ad}\, e_\alpha \cdot \mathfrak{g}_{-r\alpha} \subset \mathfrak{g}_{-(r+1)\alpha}$ for all $r \geq 1$.

Now $h'_\alpha = [e_\alpha, e_{-\alpha}]$, and so

$$\text{ad } h'_\alpha = \text{ad } e_\alpha \text{ ad } e_{-\alpha} - \text{ad } e_{-\alpha} \text{ ad } e_\alpha.$$

Thus ad $h'_\alpha \cdot \mathfrak{m} \subset \mathfrak{m}$.

We calculate the trace of ad h'_α on $\mathfrak{m}$ in two different ways. First, we have

$$\begin{aligned} \text{tr}_\mathfrak{m}(\text{ad } h'_\alpha) &= \alpha(h'_\alpha) + \dim \mathfrak{g}_{-\alpha}(-\alpha(h'_\alpha)) + \dim \mathfrak{g}_{-2\alpha}(-2\alpha(h'_\alpha)) + \cdots \\ &= \alpha(h'_\alpha)(1 - \dim \mathfrak{g}_{-\alpha} - 2\dim \mathfrak{g}_{-2\alpha} - \cdots). \end{aligned}$$

Second, we have

$$\text{tr}_\mathfrak{m}(h'_\alpha) = \text{tr}_\mathfrak{m}(\text{ad } e_\alpha \text{ ad } e_{-\alpha}) - \text{tr}_\mathfrak{m}(\text{ad } e_{-\alpha} \text{ ad } e_\alpha) = 0.$$

Thus

$$\alpha(h'_\alpha)(1 - \dim \mathfrak{g}_{-\alpha} - 2\dim \mathfrak{g}_{-2\alpha} - \cdots) = 0.$$

Now $\alpha(h'_\alpha) = \langle h'_\alpha, h'_\alpha \rangle \neq 0$, and so

$$1 - \dim \mathfrak{g}_{-\alpha} - 2\dim \mathfrak{g}_{-2\alpha} - \cdots = 0.$$

This can happen only if $\dim \mathfrak{g}_{-\alpha} = 1$ and $\dim \mathfrak{g}_{-r\alpha} = 0$ for all $r \geq 2$.

Now $\alpha \in \Phi$ if and only if $-\alpha \in \Phi$. Thus $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$. $\square$

Note that while all of the root spaces $\mathfrak{g}_\alpha$ are 1-dimensional, the space $\mathfrak{g}_0 = \mathfrak{h}$ need not be 1-dimensional.

Proposition

If $\alpha \in \Phi$ *and* $r\alpha \in \Phi$ *where* $r \in \mathbb{Z}$*, then* $r = 1$ *or* $r = -1$*.*

Proof.

From the above, we have $\dim \mathfrak{g}_{-r\alpha} = 0$ for all $r \geq 2$, that is, $-r\alpha$ is not a root.

Now $r\alpha \in \Phi$ if and only if $-r\alpha \in \Phi$. Thus only α and $-\alpha$ can be roots. □

We are now ready to examine some stronger properties of the set Φ of roots.

Let $\alpha, \beta \in \Phi$ be roots such that $\beta \neq \alpha$ and $\beta \neq -\alpha$. Then β is not an integer multiple of α.

There do, however, exist integers $p \geq 0, q \geq 0$ such that the elements

$$-p\alpha + \beta, \ldots, -\alpha + \beta, \beta, \alpha + \beta, \ldots, q\alpha + \beta$$

all lie in Φ but $-(p+1)\alpha + \beta$ and $(q+1)\alpha + \beta$ do not.

The set of roots

$$-p\alpha + \beta, \ldots, q\alpha + \beta$$

is called the **α-chain** of roots through β.

Proposition

Let α, β be roots such that $\beta \neq \alpha$ and $\beta \neq -\alpha$. Let

$$-p\alpha + \beta, \dots, \beta, \dots, q\alpha + \beta$$

be the α-chain of roots through β. Then

$$\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q.$$

Proof.

Consider the subspace $\mathfrak{m}$ of $\mathfrak{g}$ given by

$$\mathfrak{m} = \mathfrak{g}_{-p\alpha+\beta} \oplus \cdots \oplus \mathfrak{g}_{q\alpha+\beta}.$$

Recall that $h'_\alpha = [e_\alpha, e_{-\alpha}] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$.

Now β is not an integer multiple of α, and so $-(p+1)\alpha+\beta\neq 0$ and $(q+1)\alpha+\beta\neq 0$.

We have ad $e_\alpha\cdot\mathfrak{g}_{q\alpha+\beta}\subset\mathfrak{g}_{(q+1)\alpha+\beta}$. Because $(q+1)\alpha+\beta\neq 0$ and $(q+1)\alpha+\beta\notin\Phi$, we must have $\mathfrak{g}_{(q+1)\alpha+\beta}=0$.

Thus ad $e_\alpha\cdot\mathfrak{m}\subset\mathfrak{m}$. By a similar argument, we have ad $e_{-\alpha}\cdot\mathfrak{m}\subset\mathfrak{m}$, and so

$$\text{ad } h'_\alpha\cdot\mathfrak{m}=(\text{ad } e_\alpha \text{ ad } e_{-\alpha}-\text{ad } e_{-\alpha} \text{ ad } e_\alpha)\mathfrak{m}\subset\mathfrak{m}.$$

We calculate the trace of ad h'_α on $\mathfrak{m}$ in two different ways. We have

$$\text{tr}_{\mathfrak{m}}\,(\text{ad } h'_\alpha)=\sum_{i=-p}^{q}(i\alpha+\beta)\,(h'_\alpha)$$

since $\dim\mathfrak{g}_{i\alpha+\beta}=1$.

Second, we have

$$\mathrm{tr}_{\mathfrak{m}}\,(\mathrm{ad}\ h'_\alpha) = \mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\ e_\alpha\ \mathrm{ad}\ e_{-\alpha}) - \mathrm{tr}_{\mathfrak{m}}(\mathrm{ad}\ e_{-\alpha}\ \mathrm{ad}\ e_\alpha) = 0.$$

Thus

$$\sum_{i=-p}^{q} (i\alpha + \beta)\,(h'_\alpha) = 0,$$

that is,

$$\left(\frac{q(q+1)}{2} - \frac{p(p+1)}{2}\right)\alpha\,(h'_\alpha) + (p+q+1)\beta\,(h'_\alpha) = 0.$$

Since $p + q + 1 \neq 0$, this yields

$$\frac{(q-p)}{2}\left\langle h'_\alpha, h'_\alpha\right\rangle + \left\langle h'_\alpha, h'_\beta\right\rangle = 0.$$

Hence

$$\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q$$

since $\langle h'_\alpha, h'_\alpha \rangle \neq 0$. □

Corollary

If $\alpha \in \Phi$ and $\zeta\alpha \in \Phi$ where $\zeta \in \mathbb{C}$, then $\zeta = 1$ or $\zeta = -1$.

Proof.

Suppose $\zeta \neq \pm 1$ and let $\beta = \zeta\alpha$. Then $\beta(h'_\alpha) = \zeta\alpha(h'_\alpha)$, that is,

$$\langle h'_\alpha, h'_\beta \rangle = \zeta \langle h'_\alpha, h'_\alpha \rangle.$$

From the previous proposition, this yields

$$2\zeta = 2\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} = p - q.$$

Hence $2\zeta \in \mathbb{Z}$. If $\zeta \in \mathbb{Z}$, then $\zeta = \pm 1$. Thus $\zeta \notin \mathbb{Z}$. It follows that $p - q$ is odd.

The α-chain of roots through β is

$$-\left(\frac{p+q}{2}\right)\alpha, \ldots, \beta = \left(\frac{p-q}{2}\right)\alpha, \ldots, \left(\frac{p+q}{2}\right)\alpha.$$

Since $p - q$ is odd and consecutive roots differ by α, we see that all roots in the α-chain are odd multiples of $\frac{1}{2}\alpha$.

Also, $p - q \neq 0$, and so p and q cannot both be zero. Thus $p + q \neq 0$.

Because the first and last roots are negatives of one another, $\frac{1}{2}\alpha$ must lie in the α-chain. Thus $\frac{1}{2}\alpha \in \Phi$.

But $\alpha \in \Phi$, and so $2\left(\frac{1}{2}\alpha\right) \in \Phi$, a contradiction. □

Proposition

$\left\langle h'_\alpha, h'_\beta \right\rangle \in \mathbb{Q}$ for all $\alpha, \beta \in \Phi$.

Proof.

We already know that $\left\langle h'_\alpha, h'_\beta \right\rangle \in \mathbb{C}$. We also have

$$2\frac{\left\langle h'_\alpha, h'_\beta \right\rangle}{\left\langle h'_\alpha, h'_\alpha \right\rangle} \in \mathbb{Z}.$$

Thus $\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} \in \mathbb{Q}$. It is therefore sufficient to show that $\left\langle h'_\alpha, h'_\alpha \right\rangle \in \mathbb{Q}$.

We have

$$\left\langle h'_\alpha, h'_\alpha \right\rangle = \operatorname{tr}\left(\operatorname{ad} h'_\alpha \ \operatorname{ad} h'_\alpha\right) = \sum_{\beta \in \Phi} \left(\beta\left(h'_\alpha\right)\right)^2 = \sum_{\beta \in \Phi} \left\langle h'_\alpha, h'_\beta \right\rangle^2.$$

Dividing by $\langle h'_\alpha, h'_\alpha \rangle^2$, this yields

$$\frac{1}{\langle h'_\alpha, h'_\alpha \rangle} = \sum_{\beta \in \Phi} \left(\frac{\langle h'_\alpha, h'_\beta \rangle}{\langle h'_\alpha, h'_\alpha \rangle} \right)^2 \in \mathbb{Z}.$$

Hence $\langle h'_\alpha, h'_\alpha \rangle \in \mathbb{Q}$, completing the proof. □