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RESEARCH ARTICLE

Dark-dark and dark-bright soliton interactions in the two-component defocusing nonlinear Schrödinger equation

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We use the Inverse Scattering Transform machinery to construct multisoliton solutions to the 2-component defocusing nonlinear Schrödinger equation. Such solutions include dark-dark solitons, which have dark solitonic behavior in both components, as well as dark-bright soliton solutions, with one dark and one bright component. We then derive the explicit expressions of two soliton solutions for all possible cases: two dark-dark solitons, two dark-bright solitons, and one dark-dark and one dark-bright soliton. Finally, we determine the long-time asymptotic behaviors of these solutions, which allows us to obtain explicit expressions for the shifts in the phases and in the soliton centers due to the interactions.

Keywords: Inverse Scattering Transform; Nonlinear Schrödinger systems; Solitons

AMS Subject Classification: 37K15; 35K61

1. Introduction

The inverse scattering transform (IST) as a tool to solve the initial value problem for the scalar nonlinear Schrödinger (NLS) equation

\[ iq_t = q_{xx} - 2\sigma |q|^2q \]  (1.1)

(subscripts \(x\) and \(t\) denote partial differentiation throughout) has been extensively studied in the literature, both in the focusing (\(\sigma = -1\)) and in the defocusing (\(\sigma = 1\)) cases \cite{1-3}. In particular, the defocusing case with nonvanishing boundary conditions for \(q(x,t)\) as \(|x| \to \infty\) was first studied in 1973 \cite{4}; the problem was subsequently clarified and generalized in various works \cite{5-9}, and a detailed study can be found in the monograph \cite{10}.

The focusing NLS equation (i.e, Eq. (1.1) with \(\sigma = -1\)) admits the usual, bell-shaped solitons, usually referred to as “bright” solitons, that decay rapidly to zero as \(|x| \to \infty\). Instead, the defocusing NLS admits soliton solutions with nontrivial boundary conditions, the so-called dark/gray solitons, which have the form

\[ q(x,t) = q_0 e^{2\alpha^2 t} \left[ \cos \alpha + i \sin \alpha \tanh \left[ \sin \alpha q_0 (x - 2q_0 \cos \alpha t - x_0) \right] \right], \]  (1.2)

\(\alpha > 0\), \(q_0 > 0\), \(\sin \alpha q_0 > 0\), \(x_0 \geq 0\), \(t > 0\).
with \( q_0, \alpha \) and \( x_0 \) arbitrary real parameters. Such solutions satisfy the boundary conditions
\[
q(x,t) \to q_\pm(t) = q_0 e^{2i\alpha t\pm i\alpha} \quad \text{as} \quad x \to \pm \infty,
\]
and appear as localized dips of intensity \( q_0^2 \sin^2 \alpha \) on the background field \( q_0 \).

While the IST for the scalar NLS equation was developed many years ago, both with vanishing and nonvanishing boundary conditions, the basic formulation of the IST for the vector nonlinear Schrödinger (VNLS) equation:

\[
iq_t = q_{xx} - 2\sigma \|q\|^2 q,
\]

where \( q = q(x,t) \) is, in general, an \( N \)-component vector and \( \|\cdot\| \) is the standard Euclidean norm, was developed in 1974 by Manakov only in the focusing case \( (\sigma = -1) \) and with vanishing boundary conditions [11]. The IST for the defocusing vector NLS remained an almost completely open issue until 2006 (cf. [20]).

The defocusing VNLS admits dark-dark soliton solutions, which have dark solitonic behavior in all components, as well as dark-bright soliton solutions, which have (at least) one dark and one (or more than one) bright components. Therefore, the IST for defocusing VNLNLS intrinsically requires dealing with nonvanishing boundary conditions, which is notoriously more difficult. Moreover, the IST for defocusing VNLS has additional complications with respect to the scalar NLS equation (complications related to the analyticity of the eigenfunctions of the associated scattering problem) that made the development of IST elusive for over thirty years after Manakov’s pioneering work for the focusing VNLS (see [12] for some partial results on multicomponent nonlinear Schrödinger equation in the case of nonzero boundary conditions).

From a different perspective, in recent years direct methods have been applied to VNLS as a way to derive explicit bright and dark soliton solutions, see for instance Refs. [13–16] and the review article [17]. Subsequently, a method based on the Darboux transformation was presented to obtain a closed form for the multisoliton solution on a nonvanishing background of VNLS, which provided solutions that are more general than the ones derived by Hirota’s method (cf. [18] and [19]). The authors state that their method is “essentially equivalent to the IST method, but without the mathematical rigor of IST”. As stressed in [19], the IST becomes quite complicated when nonvanishing backgrounds are considered, as in the case of dark solitons. As a matter of fact, the IST for the VNLS with nonzero boundary conditions has been solved only quite recently, in [20] for the two-component case, and in [21] for the general \( N \)-component case.

It is important to point out that the understanding of the dynamics of NLS that has resulted from the development and employment of the IST is of more than mathematical interest. In fact, NLS describes, generally, the evolution of quasimonochromatic wave packets in a nonlinear medium and has been derived as a model (with experimental confirmation) in many physical contexts including, water waves, magnetic spin waves, nonlinear optics, etc.

Moreover, the correct characterization of vector soliton interactions is of great physical relevance as VNLS is a special case of a widely used model for pulse propagation in optical fibers, waveguides and waveguide arrays. Importantly, multicomponent NLS-type systems have also been the subject of renewed interest because of their applications to Bose-Einstein condensates (BECs). In single-component BECs, dark and bright solitons, forming local density suppressions and increases,
respectively, have recently been observed experimentally [22]. In two-component
BECs, the dynamics of solitons are even richer. For example, in [23], using two
distinguishable components of a BEC, the counterflow of two superfluids in a nar-
row channel has been investigated, providing the first experimental observation of
trains of dark-bright solitons.

In this work, we use the IST framework developed in [20] to derive multisoliton
dark-dark and dark-bright solutions to the two-component defocusing VNLS, in-
vestigate their interactions and obtain explicit formulas for the shifts in the phases
and in the soliton centers due to soliton interactions. These results will be relevant
from the point of view of physical applications, and they will provide a valuable
insight in the investigation of multi-component soliton interactions that we plan
to address in a future work.

2. Multisoliton solutions of defocusing VNLS via IST

It is well-known (see, for instance [24]) that the two-component defocusing VNLS
equation (1.3) with \( \sigma = 1 \) and \( N = 2 \) is associated to the Lax pair

\[
v_x = (ikJ + Q) v, \tag{2.1a}
\]

\[
v_t = \begin{pmatrix} 2ik^2 + iqv^T \mathbf{r} & -2kqv^T - iq\mathbf{r}^T \\ -2k\mathbf{r}^T + iqv & -2ik^2 - iq^T \end{pmatrix} v, \tag{2.1b}
\]

with

\[
J = \text{diag}(-1, 1, 1), \quad Q(x, t) = \begin{pmatrix} 0 & q^T \\ r & 0_{2 \times 2} \end{pmatrix}, \tag{2.2}
\]

and where: \( v(x, t, k) = (v_1(x, t, k), v_2(x, t, k), v_3(x, t, k))^T \) is the scattering eigen-
function, \( k \) is the (complex) scattering parameter, \( q(x, t) = (q_1(x, t), q_2(x, t))^T \)
and \( r(x, t) = (r_1(x, t), r_2(x, t))^T = q^*(x, t) \) are the scattering potentials, \( I_2 \)
the 2 \( \times \) 2 identity matrix, the superscript \( ^T \) denotes matrix transpose. Explicitly, the com-
patibility of the system of equations (2.1) (i.e., the equality of the mixed derivatives
of the 3-component vector \( v \) with respect to \( x \) and \( t \)), together with the constraint
\( r = q^* \), is equivalent to the requirement that \( q(x, t) \) satisfy Eq. (1.3) with \( \sigma = 1 \).

We consider solutions to VNLS with boundary conditions

\[
q(x, t) \sim q_{\pm}(t) \equiv q_0 e^{i\hat{q}_0 t} \quad \text{as } x \to \pm \infty, \tag{2.3}
\]

where \( q_{\pm} = q_0 e^{i\hat{q}_0} \), \( q_0 \) is a constant two-component complex vector, with \( ||q_0|| = \hat{q}_0 \). The asymptotic eigenvalues of the scattering problem (2.1) then turn out to be:
\( k \) and \( \pm \lambda \), with \( \lambda^2 = k^2 - \hat{q}_0^2 \).

In a similar way as for the scalar problem (e.g., see Ref. [10]), it is convenient to
introduce a uniformization variable \( z \) (global uniformizing parameter) defined by
the conformal mapping

\[
z = k + \lambda(k), \tag{2.4a}
\]

whose inverse mapping is given by

\[
k = \frac{1}{2}(z + \hat{z}^*), \quad \lambda = z - k = \frac{1}{2}(z - \hat{z}^*), \quad \hat{z} = \hat{q}_0^2 / z^*. \tag{2.3b}
\]
As shown in [20], one can then define two complete sets of eigenfunctions for the system (2.1): \(\phi_1(x, t, z), \chi(x, t, z), \phi_3(x, t, z)\) and \(\psi_1(x, t, z), \bar{\chi}(x, t, z), \psi_3(x, t, z)\), such that \(\phi_1 e^{i\lambda x}, \psi_3 e^{-i\lambda x}\) and \(\chi e^{-ikx}\) are analytic functions of the complex variable \(z\) in the upper half-plane, while \(\phi_3 e^{-i\lambda x}, \psi_1 e^{i\lambda x}\) and \(\bar{\chi} e^{-ikx}\) are analytic functions of \(z\) in the lower half-plane. The above eigenfunctions are such that

\[
\text{Wr}(\phi_1(x, z), \chi(x, z), \psi_3(x, z)) = -4g^2\lambda^2a_{11}(z) b_{33}(z) e^{ik(x)}x, \tag{2.5a}
\]

\[
\text{Wr}(\psi_1(x, z), \bar{\chi}(x, z), \phi_3(x, z)) = 4g^2\lambda^2 a_{33}(z) b_{11}(z) e^{ik(x)}x, \tag{2.5b}
\]

where \(\text{Wr}\) denotes the standard Wronskian and functions \(a_{11}(z)\) and \(b_{33}(z)\) are analytic in the upper half-plane of \(z\), while \(a_{33}(z)\) and \(b_{11}(z)\) are analytic in the lower half-plane of \(z\). Due to the symmetries of the potentials \(q\) and \(r\) (and, consequently, of the eigenfunctions), the scattering coefficients satisfy the following relations:

\[
a_{11}(\hat{z}^*) = a_{33}(z), \quad b_{11}(\hat{z}^*) = b_{33}(z), \quad b_{11}(z^*) = a_{11}(z), \quad b_{33}(z^*) = a_{33}(z). \tag{2.6a}
\]

The zeros of these scattering coefficients [where, according to (2.5) the eigenfunctions become linearly dependent] play the role of discrete eigenvalues of the scattering problem (2.1). Indeed, when one considers a pair of eigenvalues \(\zeta_n, \zeta_n^*\) on the circle \(C_0 := \{z \in \mathbb{C} : |z| = q_0\}\), such that \(a_{11}(\zeta_n) = a_{33}(\zeta_n^*) = 0\), one has:

\[
\phi_1(x, \zeta_n) = c_n \psi_3(x, \zeta_n), \quad \phi_3(x, \zeta_n^*) = \bar{c}_n \psi_3(x, \zeta_n^*), \tag{2.6a}
\]

with \(\bar{c}_n = c_n\) due to the symmetries of the eigenfunctions. Similarly, for a quartet of eigenvalues \(\{z_n, z_n^*, \hat{z}_n, \hat{z}_n^*\}\) off the circle \(C_0\), such that \(a_{11}(z_n) = b_{11}(z_n^*) = a_{33}(\hat{z}_n^*) = b_{33}(\hat{z}_n) = 0\), one has:

\[
\phi_1(x, z_n) = d_n \chi(x, \hat{z}_n^*), \quad \bar{\chi}(x, z_n^*) = \bar{d}_n \psi_3(x, \hat{z}_n), \tag{2.6b}
\]

for some (nonzero) complex coefficients \(d_n, \bar{d}_n\).

The starting point for the reconstruction of the soliton solutions via IST is given by the equations of the inverse problem for the eigenfunctions. In the pure soliton case, the eigenfunctions are given by the solution of the following linear algebraic system of equations (Eqs. (3.8) in [20], with a misprint corrected in the first equation):

\[
\psi_3(x, z) e^{-i\lambda(x)z} = \left( \begin{array}{c}
\hat{z}_n \\
n_{1+q_0}
\end{array} \right) + \sum_{n=1}^{N_1} \frac{\bar{c}_n}{|\zeta_n|} \psi_1(x, \zeta_n^*) e^{-i\lambda(z)\zeta_n^*} \tag{2.7a}
\]

\[
\psi_1(x, z) e^{i\lambda(x)z} = \left( \begin{array}{c}
z \\
n_{1+q_0}
\end{array} \right) + \sum_{n=1}^{N_1} \frac{\gamma_n}{|\zeta_n|} \psi_3(x, \zeta_n) e^{i\lambda(z)\zeta_n} \tag{2.7b}
\]

\[
\bar{\chi}(x, z) e^{-ik(x)z} = \left( \begin{array}{c}
0 \\
n_{1+q_0}
\end{array} \right) - \sum_{n=1}^{N_1} \frac{\bar{d}_n}{|\zeta_n|} \psi_1(x, z_n^*) e^{-ik(z_n^*)\zeta_n^*} \tag{2.7c}
\]
where: \( \{ \zeta_n \}_{n=1}^{N_1} \) are the zeros of the scattering coefficient \( a_{11}(z) \) on the the circle \( C_0 \) with \( \text{Im} \zeta_n > 0 \); \( \{ \xi_n \}_{n=1}^{N_2} \) are the zeros of \( a_{11}(z) \) inside the upper-half circle \( C_0 \) \( (\text{Im} \xi_n > 0) \); \( \gamma_n, \bar{\gamma}_n, \delta_n, \bar{\delta}_n \) are the so-called norming constants associated to the discrete eigenvalues \( \zeta_n, \xi_n \); \( r_\pm = q_\pm \) are the boundary conditions of the solutions, assumed as in (2.3), and \( r_\pm \) is chosen such that \( ||r_\pm|| = q_0 \) and \( r_\pm \) is orthogonal to \( r_\pm \). As we will see in the following, each \( \zeta_n \) corresponds to a dark-dark soliton in the solution; each \( z_n \), to a dark-bright soliton.

The norming constants are defined in terms of the coefficients in (2.6) as follows:

\[
\begin{align*}
\gamma_n &= \frac{\zeta_n}{\zeta_n a_{11}^t(\zeta_n)}, \\
\bar{\gamma}_n &= \frac{\bar{\zeta}_n}{\bar{\zeta}_n a_{11}^t(\bar{\zeta}_n)}, \\
\delta_n &= \frac{\bar{\zeta}_n}{z_n a_{11}(z_n)}, \\
\bar{\delta}_n &= -\frac{\bar{d}_n}{z_n b_{11}^t(z_n)}. 
\end{align*}
\]  

(2.8a, 2.8b)

Note that the time-dependence of eigenfunctions, boundary conditions and norming constants is omitted for brevity in the inverse problem. Note also that the linear system (2.7) can be simplified by taking into account the symmetry relations between the eigenfunctions, i.e. \( \psi_1(x, z) = -\psi_3(x, \bar{z}_n^+) \), which in particular for a pair of eigenvalues \( \zeta_n, \bar{\zeta}_n \) on the circle of radius \( q_0 \) implies \( \psi_1(x, \zeta_n) = -\psi_3(x, \bar{\zeta}_n) \).

The time evolution is determined by means of Eq. (2.1b), and it implies that discrete eigenvalues are time-independent, while the time dependence of the norming constants is given by (cf. [20]):

\[
\begin{align*}
\gamma_n(t) &= \gamma_n(0) e^{-i(z_n^2 - (\zeta_n^*)^2)t}, \\
\bar{\gamma}_n(t) &= \bar{\gamma}_n(0) e^{-i(\bar{z}_n^2 - (\bar{\zeta}_n)^2)t}, \\
\delta_n(t) &= \delta_n(0) e^{-i(z_n^2 + 4\bar{q}_n^2)t}, \\
\bar{\delta}_n(t) &= \bar{\delta}_n(0) e^{i((\bar{z}_n^*)^2 + 4\bar{q}_n^2)t}.
\end{align*}
\]

Equations (2.7) are the fundamental equations for the inverse scattering problem in the pure soliton case. In general, that is, when \( N_1 \neq 0 \) or \( N_2 \neq 0 \), the system is consistently closed by evaluating the first equation at \( z = \zeta_n \), for \( n = 1, \ldots, N_1 \), the second equation at \( z = \zeta_n^* \) for \( n = 1, \ldots, N_1 \) and \( z = \bar{\zeta}_n \) for \( n = 1, \ldots, N_2 \), and the last equation at \( z = \bar{\zeta}_n^* \), \( n = 1, \ldots, N_2 \). The solution \( q(x, t) = r^\ast(x, t) \) of VNLS is then reconstructed by means of the large-\( z \) expansion of the eigenfunctions in (2.7) as follows:

\[
r(x, t) = r_+(t) - i \sum_{n=1}^{N_1} \frac{\gamma_n(t)}{|\zeta_n| = q_0} \psi_3^{(dn)}(x, t, \zeta_n) e^{i\lambda(\zeta_n)x} - i \sum_{n=1}^{N_2} \delta_n(t) \chi_3^{(dn)}(x, t, \bar{\zeta}_n) e^{i\lambda(\bar{\zeta}_n)x},
\]

(2.9)

where the superscript \( ^{(dn)} \) is meant to denote the lower two components of the corresponding three-component vector eigenfunctions.

Unlike the case of initial value problems with decaying boundary conditions, and similarly to what happens for some initial-boundary value problems (see, for instance [25]), here the symmetry relations between the norming constants associated to a given eigenvalue pair/quartet might depend on all other eigenvalues. In [20] only one soliton cases were considered. Therefore, in order to obtain explicit multisoliton solutions, we derive the symmetries in the norming constants, for a reflectionless potential with an arbitrary number of dark-dark and dark-bright solitons.

First of all, in the reflectionless case, the relevant scattering coefficients, whose
zeros [assumed to be simple] correspond to the discrete eigenvalues, are given by:

\[ a_{11}(z) = \prod_{j=1}^{N_1} \frac{z - \zeta_j}{\bar{z} - \bar{\zeta}_j} \prod_{\ell=1}^{N_2} \frac{z - \bar{z}_\ell}{\bar{z} - \bar{z}_\ell}, \quad b_{11}(z) = a_{11}^*(z^*), \quad a_{33}(z) = a_{11}(q_0^2/\bar{z}). \quad (2.10) \]

Moreover, since \( a_{33}'(z) = -q_0^2 a_{11}'(q_0^2/\bar{z})/\bar{z}^2 \), we have \( a_{33}'(\zeta_n^*) = -\zeta_n a_{11}'(\zeta_n)/\zeta_n^* \), and therefore from (2.6) and (2.8) it follows

\[ \bar{\gamma}_n = -\zeta_n^* \gamma_n, \quad n = 1, \ldots, N_1. \quad (2.11a) \]

As a consequence of the symmetries in the eigenfunctions, the coefficients \( c_n \) defined in (2.6a) satisfy

\[ c_n^* = \frac{\zeta_n}{\zeta_n^*} \frac{b_{11}(z)}{a_{33}(z)} \bigg|_{z = \zeta_n^*} c_n, \]

[recall that \( \bar{c}_n = c_n \), and note that both \( b_{11}(z) \) and \( a_{33}(z) \) have simple zeros at \( z = \zeta_n^* \)]. Taking into account the explicit expressions of the scattering coefficients in (2.10), one can show that

\[ \frac{b_{11}(z)}{a_{33}(z)} \bigg|_{z = \zeta_n^*} = \prod_{j=1}^{N_1} \frac{\zeta_j^*}{\zeta_j} \prod_{\ell=1}^{N_2} \frac{(\zeta_n^* - z_j^*)(\zeta_n^* - \bar{z}_\ell^*)}{(\zeta_n^* - z_j)(\zeta_n^* - \bar{z}_\ell)}, \]

and therefore

\[ c_n^* = \frac{\zeta_n}{\zeta_n^*} \prod_{j=1}^{N_1} \frac{\zeta_j^*}{\zeta_j} \prod_{\ell=1}^{N_2} \frac{(\zeta_n^* - z_j^*)(\zeta_n^* - \bar{z}_\ell^*)}{(\zeta_n^* - z_j)(\zeta_n^* - \bar{z}_\ell)} c_n. \]

Then, from the definitions (2.8) it follows

\[ \bar{\gamma}_n^* = \frac{\zeta_n^*}{\zeta_n} \prod_{j=1}^{N_1} \frac{\zeta_j^*}{\zeta_j} \prod_{\ell=1}^{N_2} \frac{(\zeta_n^* - z_j^*)(\zeta_n^* - \bar{z}_\ell^*)}{(\zeta_n^* - z_j)(\zeta_n^* - \bar{z}_\ell)} a_{33}'(\zeta_n^*) \bar{\gamma}_n. \]

Using again the explicit expressions of the scattering coefficients in (2.10), one thus proves that the norming constants \( \bar{\gamma}_n \) also satisfy the symmetry constraints:

\[ \bar{\gamma}_n^* = -\bar{\gamma}_n, \quad n = 1, \ldots, N_1. \quad (2.11b) \]

On the other hand, the norming constants \( \delta_n, \bar{\delta}_n, \) associated to discrete eigenvalues inside the circle of radius \( q_0 \), satisfy the symmetry

\[ \bar{\delta}_n^* = -\frac{q_0^2}{z_n^2} b_{11}(q_0^2/z_n), \quad \beta_n = -2\lambda(z_n) b_{11}(q_0^2/z_n), \quad n = 1, \ldots, N_2, \quad (2.11c) \]

with

\[ b_{11}(q_0^2/z_n) = \prod_{j=1}^{N_1} \frac{\zeta_j z_n - \bar{\zeta}_j z_n}{\zeta_j z_n - \bar{\zeta}_j} \prod_{\ell=1}^{N_2} \frac{q_0^2 - z_n \bar{z}_\ell}{q_0^2 - z_n \bar{z}_\ell}. \]
An $N$ dark-bright soliton solution can then be obtained by solving the system (2.7) for $N_2 =: N$ and $N_1 = 0$ and inserting the solution into (2.9) as:

$$\mathbf{r}(x,t) = \mathbf{r}_+ e^{-2i q_0 t} \left[ 1 - \sum_{j,\ell,n=1}^N A_{j,\ell}^{-1}(x,t) \frac{\beta_j \delta_{\ell}(t) \delta_n(t) \delta_{\ell}^* z_n^*}{(z_j - z_n^*)(z_j - z_\ell)} e^{i(z_j - z_\ell)x} \right] (2.12)$$

$$+ \mathbf{r}_+ e^{-2i q_0 t} \sum_{j,\ell} \delta_j(t) \beta_{\ell} A_{j,\ell}^{-1}(x,t) e^{i z_\ell x},$$

where $A_{j,\ell}^{-1}$ are the matrix elements of $\mathbf{A}^{-1}$, with $\mathbf{A} = (A_{j,\ell})_{j,\ell=1}^N$ given by:

$$A_{j,\ell}(x,t) = \delta_{j,\ell} + \sum_{n=1}^N \frac{\beta_j \delta_n(t) \delta_{\ell} e^{i z_n^* x}}{(z_j - z_n^*)(z_j - z_\ell)} e^{i(z_j - z_\ell)x}$$

(here and in the following $\delta_{i,j}$ denotes the Kronecker delta).

Similarly, an $N$ dark-dark soliton solution is obtained can by solving the system (2.7) for $N_1 =: N$ and $N_2 = 0$ and is given by:

$$\mathbf{r}(x,t) = \mathbf{r}_+ e^{-2i q_0 t} \left[ 1 - \sum_{j,n=1}^N \gamma_n(t) \tilde{A}_{j,n}^{-1}(x,t) e^{2i \lambda(\zeta_n)x} \right], (2.13)$$

with

$$\tilde{A}_{j,n}(x,t) = \delta_{j,n} + \frac{\tilde{\gamma}_n(t)}{\xi_j - \xi_n} e^{2i \lambda(\zeta_n)x}.$$

**One dark-dark soliton.** Assuming in (2.13) $N = 1$, we obtain the expression for one dark-dark soliton solution:

$$\mathbf{q}(x,t) = \mathbf{q}_+ e^{2i q_0 t - i a} \{ \cos \alpha + i \sin \alpha \tanh [\nu(x - 2kt) - x_0] \}, (2.14)$$

where $\zeta := \zeta_1$, $\zeta = k + i \nu \equiv q_0 e^{i \alpha}$, $\tilde{\gamma} := \tilde{\gamma}_1(0)$, and

$$x_0 = \frac{1}{2} \ln \frac{\xi}{\zeta - \zeta^*} = \frac{1}{2} \ln \frac{|\tilde{\gamma}|}{2 \nu}$$

represents the center of the soliton (recall that $\tilde{\gamma}$ is purely imaginary, and in order to have a regular solution one assumes $\text{Im} \tilde{\gamma} > 0$). A dark-dark soliton is therefore a scalar function of $x,t$ (a dark soliton solution of the scalar NLS equation), multiplied by the two-component vector $\mathbf{q}_+$ (cf. Fig. A1).

**One dark-bright soliton.** Assuming in (2.12) $N = 1$, we obtain the expression for one dark-bright soliton solution:

$$\mathbf{q}(x,t) = \mathbf{q}_+ e^{2i q_0 t - i a} \{ \cos \alpha + i \sin \alpha \tanh [\nu(x - 2kt) - x_0] \} (2.15)$$

$$+ \frac{\mathbf{q}_+}{q_0} e^{2i q_0 t} e^{-ikx + i(k^2 - \nu^2)t - i \arg \delta + ia} \sin \alpha \sqrt{q_0^2 - |z|^2} \text{sech} [\nu(x - 2kt) - x_0]$$

where $z := z_1$, $z = k + i \nu \equiv |z| e^{i \alpha}$, with $|z| < q_0$, $\delta := \delta_1(0)$, $\mathbf{q}_+ = (\mathbf{r}_+)^*$ and

$$x_0 := \ln \left( \frac{q_0 |\delta|}{2 \nu} \sqrt{q_0^2 - |z|^2} \right)$$
represents the center of the soliton. If only one component of $q_\pm$ is nonzero, then
the dark and the bright parts of the soliton are separated in each component, while
travelling at the same speed $v = 2k$, with constant amplitudes (see Fig. A2).

If, on the other hand, both components of $q_\pm$ are nonzero, then the time-dependent phases cause the dark and the bright part of the soliton to interact and exhibit periodic beating (see Fig. A3).

3. Two dark-dark soliton interaction

We now consider a two dark-dark soliton solution, with $N_1 = 2$, $N_2 = 0$, $\zeta_j = k_j + i\nu_j$ and $|\zeta_j| = q_0$ for $j = 1, 2$. Let us denote by $\gamma_j$, $\bar{\gamma}_j$ the initial values of the associated norming constants, and let $\xi_j = x - 2k_j t$ for $j = 1, 2$. From Eq. (2.13) we then obtain

$$q(x,t) = q_+ e^{2i\nu_\xi t} \left\{ 1 - \frac{1}{D(x,t)} \left[ \frac{\bar{\gamma}_1}{\zeta_1} e^{-2\nu_\xi \xi_1} + \frac{\bar{\gamma}_2}{\zeta_2} e^{-2\nu_\xi \xi_2} + \frac{\bar{\gamma}_1 \bar{\gamma}_2 |\zeta_1 - \zeta_2|^2 (\zeta_1 \zeta_2 - \zeta_1^\ast \zeta_2^\ast)}{\zeta_1 \zeta_2 (\zeta_1 - \zeta_1^\ast) (\zeta_2 - \zeta_2^\ast) |\zeta_1 - \zeta_2|^2 |\zeta_1 - \zeta_2^\ast|^2} e^{-2(\nu_\xi \xi_1 + \nu_\xi \xi_2)} \right] \right\},$$

with

$$D(x,t) = 1 + \frac{\bar{\gamma}_1}{\zeta_1 - \zeta_1^\ast} e^{-2\nu_\xi \xi_1} + \frac{\bar{\gamma}_2}{\zeta_2 - \zeta_2^\ast} e^{-2\nu_\xi \xi_2} + \frac{\bar{\gamma}_1 \bar{\gamma}_2 |\zeta_1 - \zeta_2|^2 (\zeta_1 \zeta_2 - \zeta_1^\ast \zeta_2^\ast)}{\zeta_1 \zeta_2 (\zeta_1 - \zeta_1^\ast) (\zeta_2 - \zeta_2^\ast) |\zeta_1 - \zeta_2|^2 |\zeta_1 - \zeta_2^\ast|^2} e^{-2(\nu_\xi \xi_1 + \nu_\xi \xi_2)} \right\},$$

where we have also used the symmetry constraint $\bar{\gamma}_j^\ast = -\gamma_j$. Note that $D(x,t)$ is real and positive for all $x$, $t$ and all choices of soliton parameters such that $\text{Im} \bar{\gamma}_j > 0$.

The solution is plotted in Fig. A4 for some choice of the soliton parameters.

Without loss of generality, let us assume that the first soliton travels faster than the second one, i.e. let $k_1 > k_2$, so that the first soliton crosses the second one from left to right. In the limit $t \to +\infty$ and $\xi_1 = x - 2k_1 t$ fixed, Eqs. (3.1) yield

$$q(x,t) \approx q_+ e^{2i\nu_\xi t} \left\{ 1 - \frac{\bar{\gamma}_1}{\zeta_1 - \zeta_1^\ast} e^{-2\nu_\xi \xi_1} \right\}.$$

Recalling that $\bar{\gamma}_1$ is purely imaginary, and introducing

$$e^{2x_1^\ast} = \bar{\gamma}_1/(\zeta_1 - \zeta_1^\ast),$$

we can write the asymptotic behavior of the solution as

$$q(x,t) \approx q_+ e^{2i\nu_\xi t} e^{-i\zeta \xi_1} \left\{ \cos \alpha_1 + i \sin \alpha_1 \tanh[\nu_1 (x - 2k_1 t) - x_1^\ast] \right\},$$

where $\zeta_1 = k_1 + i\nu_1 = q_0 e^{i\alpha_1}$.

Similarly, the asymptotic behavior as $t \to -\infty$ with $\xi_1 = x - 2k_1 t$ fixed is given by

$$q(x,t) \approx q_+ e^{2i\nu_\xi t} \left\{ 1 - \frac{1}{D_-(\xi_1)} \left[ \frac{\zeta_2 - \zeta_2^\ast}{\zeta_2} + \frac{\bar{\gamma}_1 |\zeta_1 - \zeta_2|^2 (\zeta_1 \zeta_2 - \zeta_1^\ast \zeta_2^\ast)}{\zeta_1 \zeta_2 (\zeta_1 - \zeta_1^\ast) (\zeta_2 - \zeta_2^\ast) |\zeta_1 - \zeta_2|^2 |\zeta_1 - \zeta_2^\ast|^2} e^{-2\nu_\xi \xi_1} \right] \right\}.$$
with

$$D_-(\xi_1) = 1 + \frac{\tilde{\gamma}_1|\xi_1 - \zeta_2|^2}{(\xi_1 - \zeta_1^*)|\xi_1 - \zeta_2^*|^2} e^{-2\nu_1\xi_1},$$

and $\zeta_2 = k_2 + i\nu_2 = q_0 e^{i\alpha_2}$. In conclusion, as $t \to -\infty$

$$q(x, t) \approx q_+ e^{2ig_0^2 t} e^{-2i\alpha_2 - i\alpha_1} \{ \cos \alpha_1 + i \sin \alpha_1 \tanh [\nu_1 (x - 2k_1 t) - x^-_1] \},$$

with

$$e^{2x^-_1} = \frac{\tilde{\gamma}_1 |\xi_1 - \zeta_2|^2}{(\xi_1 - \zeta_1^*)|\xi_1 - \zeta_2^*|^2}.$$  \hspace{1cm} (3.3)

Comparing with (3.2), we obtain the expression for the phase shift

$$\Delta x_1 = x^+_1 - x^-_1 = \ln \frac{|\xi_1 - \zeta_2|}{|\xi_1 - \zeta_2|}.$$ \hspace{1cm} (3.4)

The corresponding expressions for the asymptotic behavior along the direction of the second soliton are obtained by interchanging indices $1 \leftrightarrow 2$ and by switching $t \to -\infty$ with $t \to +\infty$. As a consequence, one also has $\Delta x_2 = -\Delta x_1$.

4. Interaction between a dark-bright and a dark-dark soliton

Let us now consider a solution to the linear system (2.7) with $N_1 = N_2 = 1$, which corresponds to a dark-dark + dark-bright soliton solution. Let $k = k_b + i\nu_b$, with $\nu_b > 0$ and $|z| < q_0$, and $\zeta = k_d + i\nu_d$ with $\nu_d = \sqrt{q_0^2 - k_d^2} > 0$, denote the corresponding discrete eigenvalues, and $\delta, \bar{\delta}$ and $\gamma, \bar{\gamma}$ the initial values of the norming constants. Let us also introduce $\xi_b = x - 2k_b t$ and $\xi_d = x - 2k_d t$ to denote the coordinates in the reference frame of each soliton. Using the symmetry relation (2.11e), after some algebraic manipulation we find the expression of the dark-dark+dark-bright soliton solution:

$$q(x, t) = q_+ e^{2ig_0^2 t} \left\{ 1 + \frac{1}{D(x, t)} \left[ -\frac{\gamma}{\zeta - \zeta^*} e^{-2\nu_0\xi_0} + \frac{q_0^2 \bar{\nu}_0 |\delta|^2 (q_0^2 - |z|^2)|z - \zeta|^2}{z(z - \zeta^*)} e^{-2\nu_0\xi_0} \right. \right.$$  \hspace{1cm} (4.1a)

$$+ \frac{q_0^2 \gamma |\delta|^2 (q_0^2 - |z|^2)(\zeta - \zeta|^*)|z - \zeta|^4}{z(z - \zeta^*)^2(\zeta - \zeta^*)} e^{-2(\nu_0\xi_0 + \nu_0\xi_0)} \}$$

$$+ q_+^2 e^{2ig_0^2 t} \gamma (q_0^2 - |z|^2) \zeta (\zeta^* - z^*) \frac{e^{-\nu_0\xi_0-i\nu_0 k_d+i(k_0^2 - k_0^2) t}}{z^* \zeta^* (\zeta - z^*)} \left\{ 1 + \frac{1}{D(x, t)} \left[ -\frac{\gamma}{\zeta - \zeta^*} e^{-2\nu_0\xi_0} \right. \right.$$  \hspace{1cm} (4.1b)

$$+ \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2)|z - \zeta|^2}{(z - \zeta^*)^2} e^{-2\nu_0\xi_0} + \frac{q_0^2 \gamma |\delta|^2 (q_0^2 - |z|^2)|z - \zeta|^4}{(z - \zeta^*)^2(\zeta - \zeta^*)} e^{-2(\nu_0\xi_0 + \nu_0\xi_0)} \right\}$$

with

$$D(x, t) = 1 + \frac{\gamma}{\zeta - \zeta^*} e^{-2\nu_0\xi_0} + \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2)|z - \zeta|^2}{(z - \zeta^*)^2} e^{-2\nu_0\xi_0}. \hspace{1cm} (4.1b)$$
It is immediate to check that $D(x,t)$ is real and positive for any choice of the soliton parameters. The solution is plotted in Fig. A5 when both components of $q_\pm$ are nonzero, and in Fig. A6 when one component in $q_\pm$ is zero.

Without loss of generality, let us assume that the dark-dark soliton travels faster than the dark-bright soliton, i.e. let $k_d > k_b$. In the limit $t \to +\infty$ and $\xi_d = x - 2k_d t$ fixed, Eqs. (4.1) yield

$$ q(x,t) \simeq q_+ e^{2 i q_d t} \left[ 1 - \frac{\tilde{\gamma}}{\zeta - \zeta^*} e^{-2\nu_\alpha x + 4k_b \nu_\delta t} \right], $$

i.e.,

$$ q(x,t) \simeq q_+ e^{2 i q_d t - i \alpha_d} \{ \cos \alpha_d + i \sin \alpha_d \tanh[\nu_d(x - 2k_d t) - x_d^+] \} \quad (4.2) $$

with $\zeta = q_0 e^{i \alpha_d}$ and $e^{2x_d^+} = \tilde{\gamma}/(\zeta - \zeta^*)$.

Similarly, as $t \to -\infty$ with $\xi_d$ fixed, (4.1) yield

$$ q(x,t) \simeq q_+ e^{2 i q_d t} \left\{ 1 - \frac{1}{2} \left( z - z^* \right) + \frac{\tilde{\gamma} (z - \zeta^*)}{(\xi - \zeta^*)} e^{-2\nu_\alpha x + 4k_b \nu_\delta t} \right\}. $$

Introducing

$$ e^{2x_d^-} = \frac{\tilde{\gamma}|z - \zeta|^2}{(\zeta - \zeta^*)(z - \zeta^*|^2), $$

we finally obtain

$$ q(x,t) = q_+ e^{2 i q_d t - i \alpha_d - 2i \alpha_b} \{ \cos \alpha_d + i \sin \alpha_d \tanh[\nu_d(x - 2k_d t) - x_d^-] \}, \quad (4.3) $$

where $z = |z|e^{i \alpha_b}$. Therefore, for the phase shift of the dark-dark soliton one has

$$ \Delta x_d = \frac{|z - \zeta^*|}{|z - \zeta|}. \quad (4.4) $$

Now, fixing $\xi_b = x - 2k_b t$, as $t \to -\infty$ in (4.1) we have:

$$ q(x,t) \simeq q_+ e^{2 i q_d t} \left\{ 1 + \frac{1}{D_-(\xi_b)} \frac{\tilde{\gamma}^2 |\xi|^2 (q_0^2 - |z|^2)}{z(z - z^*)|z - \zeta^*|^2} e^{-2\nu_\alpha \xi_b} \right\} + q_+ e^{2 i q_d t} \frac{1}{D_-(\xi_b)} \frac{\tilde{\gamma}^2 (q_0^2 - |z|^2) \zeta (\zeta^* - z^*)}{z^* \zeta^* (\zeta - z^*)} e^{-\nu_\alpha \xi_b - ik_b x - i(k_b^2 - \nu_\delta^2) t} $$

where

$$ D_-(\xi_b) = 1 - \frac{\tilde{\gamma}^2 |\xi|^2 (q_0^2 - |z|^2)}{z(z - z^*)|z - \zeta^*|^2} e^{-2\nu_\alpha \xi_b}. $$
The above expression can be written as
\[
q(x, t) \simeq q_+ e^{2i\omega t - i\phi_2 - \alpha_2 x} \left\{ \cos \alpha \delta + i \sin \alpha \delta \tanh [\nu (x - 2k b t - x^-)] \right\}
\]
\[
+ \frac{q_+ e^{2i\omega t}}{q_0} e^{-ik_x x + i(k_2 - \nu_2) t - i(\phi_1 - \alpha + \eta - \eta_2)} \sin \alpha \delta \sqrt{q_0^2 - |z|^2 \sech [\nu (x - 2k b t - x^-)]},
\]
where we introduced
\[
e^{-2x^-} = \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2) |z - \delta|^2}{4 \nu_0^2 |z - \delta|^2},
\]
and
\[
e^{-2x^+} = \frac{z - \delta}{|z - \delta|}, \quad e^{-2x^*} = \frac{z - \delta^*}{|z - \delta|}, \quad \varphi_1 = \arg \delta + \theta_+ , \quad \varphi_2 = -\theta_+.
\]
Similarly, as \( t \rightarrow +\infty \) Eqs. (4.1) yield
\[
q(x, t) \simeq q_+ e^{2i\omega t} \left\{ 1 + \frac{1}{\zeta D_+ (\xi_b)} \left[ (\zeta^* - \zeta) + \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2) (\zeta - \zeta^*) |z - \zeta|^4}{z(z - \zeta^*)^2 (z - \zeta^*)^4} e^{-2\nu \xi_b} \right] \right\}
\]
\[
+ \frac{q_+ e^{2i\omega t}}{\zeta^* (\zeta - \zeta^*)^2} \frac{\delta^2 \zeta (q_0^2 - |z|^2) (\zeta^* - \zeta^*)^2}{z(z - \zeta^*)^2 (z - \zeta^*)^4} e^{-2\nu \xi_b} \frac{D_+ (\xi_b)}{D_+ (\xi_b)}
\]
where
\[
D_+ (\xi_b) = 1 - \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2) |z - \zeta|^4}{(z - \zeta^*)^2 (z - \zeta^*)^4} e^{-2\nu \xi_b}.
\]
Introducing
\[
e^{-2x^+} = \frac{q_0^2 |\delta|^2 (q_0^2 - |z|^2) |z - \zeta|^4}{4 \nu_0^2 |z - \zeta^*|^4}
\]
we finally obtain
\[
q(x, t) \simeq q_+ e^{2i\omega t - 2i\alpha_2 - i\phi_2 - \alpha_2 x} \left\{ \cos \alpha \delta + i \sin \alpha \delta \tanh [\nu (x - 2k b t - x^-)] \right\}
\]
\[
+ \frac{q_+ e^{2i\omega t}}{q_0} \sin \alpha \delta \sqrt{q_0^2 - |z|^2 e^{-ik_x x + i(k_2 - \nu_2) t - 2i(\eta_1 - \alpha) - i(\phi_1 - \alpha)} \sech [\nu (x - 2k b t - x^-)]}.
\]
The comparison between the two asymptotic limits finally yields
\[
\Delta x_b = x^+_b - x^-_b = \ln \left| \frac{z - \zeta}{z - \zeta^*} \right|,
\]
with \( \Delta x_d = -\Delta x_b \), as expected.

5. Two dark-bright soliton interaction

Finally, we consider a 2 dark-bright soliton solution, obtained from Eqs. (2.12) with \( z_j = k_j + i\nu_j \) and \( |z_j| < q_0 \) for \( j = 1, 2 \). As before, let us denote by \( \delta_j \) the initial
values of the associated norming constants, and let $\xi_j = x - 2k_j t$ for $j = 1, 2$. Using
the symmetry relations derived from (2.11c), i.e.

$$\tilde{\delta}_1 = -\frac{q_0^2(q_0^2 - |z_1|^2)(q_0^2 - z_1^* z_2)}{(z_1^*)^2(q_0^2 - z_1^* z_2)} \delta_1^*, \quad \tilde{\delta}_2 = -\frac{q_0^2(q_0^2 - |z_2|^2)(q_0^2 - z_1^* z_2)}{(z_2^*)^2(q_0^2 - z_1^* z_2)} \delta_2^*, $$

after some tedious but straightforward algebraic simplifications, we obtain

$$q(x, t) = q_* e^{2iq_0^2 t} \left\{ 1 + \frac{1}{D(x, t)} \left[ \frac{\delta_1(z_1^*)^2}{q_0^2 - z_1 z_2} \left( \frac{z_1^* z_2}{z_1(z_1^* - z_1)} e^{-2\nu_1 \xi_1} - \frac{q_0^2 - |z_2|^2}{z_2(z_2^* - z_2)} e^{i(z_2 - z_2^*)/(x - (z_2 + z_2^*) t)} \right) \right. \\
+ \frac{\delta_2(z_2^*)^2}{q_0^2 - z_1 z_2} \left( \frac{z_1^* z_2}{z_2(z_2^* - z_2)} e^{-2\nu_2 \xi_2} - \frac{q_0^2 - |z_1|^2}{z_1(z_1^* - z_1)} e^{i(z_1 - z_1^*)/(x - (z_1 + z_1^*) t)} \right) \right. \\
\left. \left. + \frac{\delta_0^2(q_0^2 - |z_1|^2)(q_0^2 - |z_2|^2)(q_0^2 - z_1 z_2)}{16q_0^4 |z_1|^2 |z_2|^2 (q_0^2 - z_1 z_2^2)^2 (z_1^* - z_1^*)^2} e^{-2\nu_1 \xi_1 - 2\nu_2 \xi_2} \right) \right\}, \quad (5.1a)$$

where the expression of denominator is given by

$$D(x, t) = 1 - \frac{\delta_1(z_1^*)^2}{q_0^2 - z_1 z_2} \left( \frac{z_1^* z_2}{z_1(z_1^* - z_1)} e^{-2\nu_1 \xi_1} - \frac{q_0^2 - |z_2|^2}{z_2(z_2^* - z_2)} e^{i(z_2 - z_2^*)/(x - (z_2 + z_2^*) t)} \right) $$

$$+ \frac{\delta_2(z_2^*)^2}{q_0^2 - z_1 z_2} \left( \frac{z_1^* z_2}{z_2(z_2^* - z_2)} e^{-2\nu_2 \xi_2} - \frac{q_0^2 - |z_1|^2}{z_1(z_1^* - z_1)} e^{i(z_1 - z_1^*)/(x - (z_1 + z_1^*) t)} \right) $$

$$+ \frac{\delta_0^4 |z_1|^2 |z_2|^2}{16q_0^4 |z_1|^2 |z_2|^2 (q_0^2 - z_1 z_2^2)^2 (z_1^* - z_1^*)^2} e^{-2\nu_1 \xi_1 - 2\nu_2 \xi_2} .$$

In the appendix we show that $D(x, t) \geq 1$ for all $x, t$, and therefore the 2-soliton solution is regular for any choice of parameters. The solution is plotted in Fig. A7.

Without loss of generality, let us assume that the first soliton travels faster than second soliton, i.e. let $k_1 > k_2$, so that the first soliton crosses the second one from left to right. In the limit $t \to +\infty$ and $\xi_1 = x - 2k_1 t$ fixed, Eqs. (5.1) yield

$$q(x, t) \simeq q_* e^{2iq_0^2 t} \left\{ 1 + \frac{q_0^2 |z_1|^2 (q_0^2 - |z_1|^2)(q_0^2 - z_1 z_2^2)}{z_1(z_1^* - z_1)(q_0^2 - z_1 z_2^2)^2} e^{-2\nu_1 \xi_1} \right\}$$

$$- q_1^* e^{2iq_0^2 t} \frac{q_0^2 |z_1|^2}{q_0^2} e^{-ik_2 x + i(k_1^2 - k_2^2) t} \frac{e^{-\nu_1 \xi_1}}{D_+(\xi_1)},$$

where

$$D_+(\xi_1) = 1 + \frac{q_0^2 |z_1|^2 (q_0^2 - |z_1|^2)(q_0^2 - z_1 z_2^2)}{4q_0^2 |z_1|^2 |z_2|^2 (q_0^2 - z_1 z_2^2)^2} e^{-2\nu_1 \xi_1}.$$
Taking \( \mathbf{q}_+ = (q_0, 0)^T \) we obtain for the two components of \( \mathbf{q}(x, t) \):

\[
q_1(x, t) = q_0 e^{2iq_0t - i\varphi_1 - i\alpha_1} \left\{ \cos \alpha_1 + i \sin \alpha_1 \tanh[\nu_1(x - 2k_1t) - x_1^+] \right\},
\]

\[
q_2(x, t) = e^{-ik_1x + i[2iq_0^2 + k_1^2 - \nu_1^2](\alpha_1 - \varphi_1 - \eta_1 + \nu_2)} \sin \alpha_1 \sqrt{q_0^2 - |z_1|^2} \sech[\nu_1(x - 2k_1t) - x_1^+] \]

where

\[
\begin{align*}
q_1 &= |z_1| e^{i\alpha_1} = \alpha_1 + iv_1, & \varphi_1 := \arg \delta_1 + \theta_1, & \varphi_2 := -\theta_1, \\
\end{align*}
\]

\[
e^{im_1} := (q_0^2 - z_1 z_2^*)/|q_0^2 - z_1 z_2^*|, & e^{m_2} := (q_0^2 - z_1 z_2)/|q_0^2 - z_1 z_2|,
\]

and

\[
e^{2x_1^+} = \frac{q_0^2}{4\nu_1^2} \frac{q_0^2 - |z_1|^2}{|q_0^2 - z_1 z_2^*|^2} \]. \quad (5.3)
\]

Similarly, the long-time asymptotic behavior as \( t \to -\infty \) along the direction of the first soliton, i.e., with \( \xi_1 = x - 2k_1t \) fixed, yields

\[
\begin{align*}
\mathbf{q}(x, t) &\simeq \mathbf{q}_+ e^{2iq_0^2t} \left\{ 1 - \frac{1}{D_-(\xi_1)} \left[ \frac{2i\nu_2}{2z_2} - \frac{q_0^2 |\delta_1|^2 q_0^2 - |z_1|^2}{4\nu_1^2 z_1 z_2^*} \right] e^{-2\nu_1 \xi_1} \right\} \\
& \quad - \frac{q_+ e^{2iq_0^2t} \cdot \bar{\delta}_1 z_1^* z_2 (q_0^2 - z_1 z_2^*) (z_1^* - z_2^*) e^{-\nu_1 \xi_1 - ik_1x + i(k_1^2 - \nu_1^2)t}}{D_-(\xi_1) z_2^* (q_0^2 - z_1 z_2) (z_1^* - z_2^*)} \]
\]

with

\[
D_-(\xi_1) = 1 + \frac{q_0^2 |\delta_1|^2 (q_0^2 - |z_1|^2) |z_1 - z_2|}{4\nu_1^2 |z_1^* - z_2^*|^2} e^{-2\nu_1 \xi_1}. \]

As before, when the dark and bright components are separated, we then obtain

\[
q_1(x, t) \simeq q_0 e^{2iq_0^2t - 2i\alpha_2 - i\varphi_2 - i\alpha_1} \left\{ \cos \alpha_1 + i \sin \alpha_1 \tanh[\nu_1(x - 2k_1t) - x_1^-] \right\},
\]

\[
q_2(x, t) \simeq e^{2iq_0^2t + i(\alpha_1 - \varphi_1) + 2i(\alpha_2 + \mu_1 - \mu_2) - ik_1x + i(k_1^2 - \nu_1^2)t} \sin \alpha_1 \sqrt{q_0^2 - |z_1|^2} \sech[\nu_1(x - 2k_1t) - x_1^-]
\]

with

\[
\begin{align*}
\varphi_2 &= \arg \delta_1 + \theta_1, & \nu_1 := |z_2| \cos \alpha_2, & \nu_2 := |z_2| \sin \alpha_2, \\
\end{align*}
\]

\[
e^{i\mu_1} := (z_1 - z_2)/|z_1 - z_2|, & e^{i\mu_2} := (z_2^* - z_1)/|z_2^* - z_1|,
\]

and

\[
e^{2x_1^-} = \frac{q_0^2 |\delta_1|^2 (q_0^2 - |z_1|^2) |z_1 - z_2|}{4\nu_1^2 |z_1^* - z_2^*|^2}. \quad (5.5)
\]

Comparing the two asymptotic behaviors, we obtain for the phase shift of the soliton the following expression:

\[
\Delta x_1 := x_1^+ - x_1^- = \ln \frac{|q_0^2 - z_1 z_2^*| |z_1^* - z_2^2|^2}{|q_0^2 - z_1 z_2| |z_1 - z_2|^2}. \quad (5.6)
\]

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The corresponding expressions for the asymptotic behavior along the direction of
the second soliton are obtained by interchanging indices $1 \leftrightarrow 2$ and by switching $t \to -\infty$ with $t \to +\infty$. As a consequence, one also has $\Delta x_2 = -\Delta x_1$.

We note that the expressions for the position shifts of two dark-dark solitons, i.e.
Eq. (3.4), or one dark-dark and one dark-bright soliton, i.e., Eqs. (4.4) and (4.5),
can be obtained as a limit of Eq. (5.6) for $|z_j| \to q_0$ for either $j = 1$ or $j = 2$.

6. Conclusion

In this paper we have used the IST machinery developed in [20] to construct
and investigate multisoliton solutions to the 2-component defocusing nonlinear
Schrödinger equation. Such solutions include dark-dark solitons, which have dark
solitonic behavior in both components, as well as dark-bright soliton solutions, with
one dark and one bright component. We have derived the explicit expressions of
two soliton solutions for all possible cases: two dark-dark solitons, two dark-bright
solitons, and one dark-dark and one dark-bright soliton. The long-time asymptotic
behavior of these solutions before and after any interactions allowed us to obtain
explicit expressions for the shifts in the phases and in the soliton centers due to
the interactions.

In view of the growing interest in vector NLS soliton interactions due to the
recent experimental observations of dark-dark and dark-bright soliton trains in
BECs, the results presented in this paper are likely to be useful from the point of
view of physical applications. This work will also provide a valuable insight in the
investigation of interaction of vector solitons with more than 2-components, and
in extending the IST to more general nonzero boundary conditions, which we plan
to address in the future.

Appendix A. Regularity of two dark-bright soliton solution

Let us consider the expression for $D(x,t)$ in (5.1b), which we can rewrite using the
symmetries in the norming constants as follows:

$$D(x,t) = 1 + q_0^4 |\delta_1|^2 |\delta_2|^2 \frac{(q_0^2 - |z_1|^2)}{4\nu_1^2 4\nu_2^2 |q_0^2 - z_1 z_2|^2} \frac{q_0^2 - z_1 z_2}{|q_0^2 - z_1 z_2|^2} e^{-2(\nu_1 \xi_1 + \nu_2 \xi_2)}$$

$$+ q_0^2 |\delta_1|^2 \frac{(q_0^2 - |z_1|^2)}{4\nu_1^2 |q_0^2 - z_1 z_2|^2} e^{-2\nu_1 \xi_1} + q_0^2 |\delta_2|^2 \frac{(q_0^2 - |z_2|^2)}{4\nu_2^2 |q_0^2 - z_1 z_2|^2} e^{-2\nu_2 \xi_2}$$

$$- 2q_0^2(q_0^2 - |z_1|^2)(q_0^2 - |z_2|^2) e^{-2(\nu_1 \xi_1 + \nu_2 \xi_2)} \times$$

$$\frac{1}{|q_0^2 - z_1 z_2|^2} \times \Re \left\{ \frac{\delta_1 \delta_2 (q_0^2 - z_1 z_2)}{(z_2^2 - z_1^2)^2} e^{i(k_1 - k_2)x + i(k_2^2 - k_1^2) t - i(k_2^2 - k_1^2) t'} \right\}$$

where $\xi_j = x - 2k_j t$ for $j = 1, 2$. In this form, it is clear that $D(x,t)$ is real (the
first 4 terms being real and positive, when both eigenvalues $z_1, z_2$ are inside the
circle $C_0$). We show next that $D(x,t) \geq 1$ for all $x, t$ and any choice of the soliton
parameters. Consider the last three terms in (A1), introducing

\[ A_1 = q_0 \frac{|\delta_1|}{2\nu_1 |q_0^2 - z_1^2|} \sqrt{q_0^2 - |z_1|^2} e^{-\nu_1 \xi_1}, \quad A_2 = q_0 \frac{|\delta_2|}{2\nu_2 |q_0^2 - z_1^2|} \sqrt{q_0^2 - |z_2|^2} e^{-\nu_2 \xi_2} \]

and noticing that the terms in the second row of (A1) are simply \( A_1^2 + A_2^2 \). Then we observe that

\[ 0 \leq (A_1 - A_2)^2 = A_1^2 + A_2^2 - 2 \frac{q_0^2 |\delta_1| |\delta_2| |q_0^2 - z_1^2 z_2| \sqrt{q_0^2 - |z_1|^2} \sqrt{q_0^2 - |z_2|^2}}{4\nu_1 \nu_2 |q_0^2 - z_1 z_2|^2} e^{-\nu_1 \xi_1 - \nu_2 \xi_2}. \]

Since

\[ |z_1 - z_2|^2 \geq \left( \frac{q_0^2 - |z_1|^2}{q_0^2 - |z_2|^2} \right) \left( q_0^2 - |z_2|^2 \right), \]

it then follows that

\[ 0 \leq (A_1 - A_2)^2 \leq A_1^2 + A_2^2 - 2 \frac{q_0^2 |\delta_1| |\delta_2| |q_0^2 - z_1^2 z_2| \left( q_0^2 - |z_1|^2 \right) \left( q_0^2 - |z_2|^2 \right)}{4\nu_1 \nu_2 |q_0^2 - z_1 z_2|^2} e^{-\nu_1 \xi_1 - \nu_2 \xi_2}. \]

Moreover, \( |z_1 - z_2|^2 = (k_1 - k_2)^2 + (\nu_1 + \nu_2)^2 \geq (\nu_1 + \nu_2)^2 \geq 4\nu_1 \nu_2, \) and therefore

\[ 0 \leq (A_1 - A_2)^2 \leq A_1^2 + A_2^2 - 2 \frac{q_0^2 |\delta_1| |\delta_2| |q_0^2 - z_1^2 z_2| \left( q_0^2 - |z_1|^2 \right) \left( q_0^2 - |z_2|^2 \right)}{4\nu_1 \nu_2 |q_0^2 - z_1 z_2|^2} e^{-\nu_1 \xi_1 - \nu_2 \xi_2} \]

\[ \leq A_1^2 + A_2^2 - 2 \frac{q_0^2 |\delta_1| |\delta_2| |q_0^2 - z_1^2 z_2| \left( q_0^2 - |z_1|^2 \right) \left( q_0^2 - |z_2|^2 \right)}{4\nu_1 \nu_2 |q_0^2 - z_1 z_2|^2} e^{-\nu_1 \xi_1 - \nu_2 \xi_2} \]

\[ \leq A_1^2 + A_2^2 - 2q_0^2 e^{-\nu_1 \xi_1 - \nu_2 \xi_2} \frac{\left( q_0^2 - |z_1|^2 \right) \left( q_0^2 - |z_2|^2 \right)}{|q_0^2 - z_1 z_2|^2} \times \]

\[ \times \text{Re} \left\{ \delta_1 \delta_2^* \frac{q_0^2 - z_1 z_2^2}{z_1^2 - z_2^2} e^{i(k_1 - k_2)x + i(k_2^2 - \nu_2^2)t - i(k_1^2 - \nu_1^2)t} \right\}. \]

The right-hand-side is exactly the sum of the last three terms in \( D \) in (A1), thus proving that \( D(x, t) \geq 1 \).

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References


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Figure A1. One dark-dark soliton: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $\zeta = 1/2 + i$, $q_+ = (1, 3/4)^T$, $\bar{\gamma} = 5i$.

Figure A2. One dark-bright soliton: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $z = .25 + .5i$, $\delta = 2 + i$ and $q_+ = (1, 0)^T$.

Figure A3. One dark-bright soliton: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $z = .25 + .5i$, $\delta = 2 + i$ and $q_+ = (1, 1)^T$.

Figure A4. Two dark-dark solitons: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $k_1 = 1$, $k_2 = 1/2$, $\gamma_1 = i$, $\gamma_2 = 2i$ and $q_+ = (1, 3/4)^T$. 
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Figure A5. One dark-dark + one dark-bright solitons: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $k_d = 0.5$, $k_b = 0.25$, $\nu_b = 1$, $\bar{\gamma} = i$, $\bar{\delta} = 2 + 5i$ and $q_+ = (1,1)^T$.

Figure A6. One dark-dark + one dark-bright solitons: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $k_d = 0.5$, $k_b = 0.25$, $\nu_b = 1$, $\bar{\gamma} = i$, $\bar{\delta} = 2 + 5i$ and $q_+ = (1,0)^T$.

Figure A7. Two dark-bright solitons: $|q_j(x,t)|^2$ is plotted for $j = 1$ (left) and $j = 2$ (right). The soliton parameters are: $z_1 = 60/99 + .7i$, $z_2 = -65/100 + .6i$, $\delta_1 = 1 + i$, $\delta_2 = 1 + 2i$ and $q_+ = (1,0)^T$. 