Math 3130 Fall 2008

Solution to Mid-term No. 1

I. A C B

II. By elementary row operations

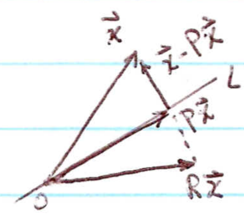
$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 3 & -6 & -6 & 8 & 2 \\ -2 & 4 & 5 & -5 & -1 \end{bmatrix} \xrightarrow[\mathrm{I}\times 2+\mathrm{III}]{\mathrm{I}\times(-3)+\mathrm{II}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & -3 & -1 & 2 \\ 0 & 0 & 3 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{\mathrm{II}\times(-\frac{1}{3})} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 3 & 1 & -1 \end{bmatrix} \xrightarrow{\mathrm{II}\times(-3)+\mathrm{III}} \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The last eqn. becomes $0=1$, hence the system has no solution

III. 1. The unit vector along $\vec{a}$ is $\vec{u} = [-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}]^T$

So the matrix is $P = \begin{bmatrix} u_1^2 & u_1u_2 & u_1u_3 \\ u_1u_2 & u_2^2 & u_2u_3 \\ u_1u_3 & u_2u_3 & u_3^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{4}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{4}{9} & \frac{4}{9} \end{bmatrix}$



2. For any $\vec{x} \in \mathbb{R}^3$, the projection onto L is $P\vec{x}$

So the reflection about L is $\vec{x} - 2(\vec{x} - P\vec{x}) = (2P - I_3)\vec{x}$,

and the matrix is $R = 2P - I_3 = \begin{bmatrix} -\frac{7}{9} & -\frac{4}{9} & -\frac{4}{9} \\ -\frac{4}{9} & -\frac{1}{9} & \frac{8}{9} \\ -\frac{4}{9} & \frac{8}{9} & -\frac{1}{9} \end{bmatrix}$

IV. By elementary row operations,

$$[A\ I_3] = \begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -2 & 1 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\mathrm{I}\leftrightarrow\mathrm{II}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & -4 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\mathrm{I}\times 2+\mathrm{III}}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{bmatrix} \xrightarrow{\mathrm{II}\times(-1)+\mathrm{III}} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix} \xrightarrow[\mathrm{III}\times(-2)+\mathrm{I}]{\mathrm{III}+\mathrm{II}} \begin{bmatrix} 1 & 0 & 0 & 2 & -3 & -2 \\ 0 & 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{bmatrix}$$

So the rref of A is I_3, A is invertible, and $A^{-1} = \begin{bmatrix} 2 & -3 & -2 \\ 0 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix}$.

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Solution to Mid-term Exam No. 2

I. A C C D B

II. By elementary row operations,

$$A=\begin{bmatrix}1&0&2&4\\0&1&-3&-1\\3&4&-6&8\end{bmatrix}\begin{matrix}\\ \\ -3(\mathrm{I})\end{matrix} \longrightarrow \begin{bmatrix}1&0&2&4\\0&1&-3&-1\\0&4&-12&-4\end{bmatrix}\begin{matrix}\\ \\ -4(\mathrm{II})\end{matrix} \longrightarrow \begin{bmatrix}1&0&2&4\\0&1&-3&-1\\0&0&0&0\end{bmatrix}=\mathrm{rref}(A).$$

Suppose $A=[\vec{a}_1\ \vec{a}_2\ \vec{a}_3\ \vec{a}_4]$ and $\mathrm{rref}(A)=[\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4]$, then $\vec{b}_1, \vec{b}_2$ are linear independent and $\vec{b}_3=2\vec{b}_1-3\vec{b}_2$, $\vec{b}_4=4\vec{b}_1-\vec{b}_2$.

So $\vec{a}_1, \vec{a}_2$ are linear independent and $\vec{a}_3=2\vec{a}_1-3\vec{a}_2$, $\vec{a}_4=4\vec{a}_1-\vec{a}_2$.

Therefore $\left(\begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}0\\1\\4\end{bmatrix}\right)$ is a basis of $\mathrm{im}(A)$.

Since $2\vec{a}_1-3\vec{a}_2-\vec{a}_3=\vec{0}$, $4\vec{a}_1-\vec{a}_2-\vec{a}_4=\vec{0}$, $\left(\begin{bmatrix}2\\-3\\-1\\0\end{bmatrix}, \begin{bmatrix}4\\-1\\0\\-1\end{bmatrix}\right)$ is a basis of $\ker(A)$.

III. 1. $P^2\in V$ if and only if $P^2=aI_2+bP$ has a solution (a,b).

$P^2=aI_2+bP$ is equivalent to $\begin{bmatrix}1&9\\0&4\end{bmatrix}=a\begin{bmatrix}1&0\\0&1\end{bmatrix}+b\begin{bmatrix}1&3\\0&2\end{bmatrix}$, i.e. $\begin{cases}a+b=1\\a+2b=4\\3b=9\end{cases}$,

which has a solution $(a,b)=(-2,3)$. So $P^2\in V$ and $[P^2]_{\mathfrak{B}}=\begin{bmatrix}-2\\3\end{bmatrix}$.

2. Using $P^2=-2I_2+3P$, we get $T(I_2)=P$ and $T(P)=P^2=-2I_2+3P$.

So $[T(I_2)]_{\mathfrak{B}}=\begin{bmatrix}0\\1\end{bmatrix}$ and $[T(P)]_{\mathfrak{B}}=\begin{bmatrix}-2\\3\end{bmatrix}$. Hence $[T]_{\mathfrak{B}}=\begin{bmatrix}0&-2\\1&3\end{bmatrix}$.

T is an isomorphism because $[T]_{\mathfrak{B}}$ is invertible.

(Another reason for this is that the inverse of T can be constructed explicitly as $T^{-1}(M)=MP^{-1}$, since P is invertible.)

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Solution to Mid-term Exam No. 3

I. D C A D

II. 1. $\vec{v}_1 = [1\ 1\ {-1}\ 1]^T$, $\vec{v}_2 = [-1\ 0\ 1\ 0]^T$. By the Gram-Schmidt process,

$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\vec{v}_1 = [\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}]^T$, $\vec{v}_2^{\,/\!/} = (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = [-\frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}]^T$,

$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\,/\!/} = [-\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]^T$, $\vec{u}_2 = \frac{1}{\|\vec{v}_2^{\perp}\|}\vec{v}_2^{\perp} = [-\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]^T$.

So $\vec{u}_1 = [\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}]^T$, $\vec{u}_2 = [-\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]^T$ form an orthonormal basis of W

2. Since $\vec{v}_1 = \|\vec{v}_1\|\vec{u}_1 = 2\vec{u}_1$, $\vec{v}_2 = \vec{v}_2^{\,/\!/} + \vec{v}_2^{\perp} = -\vec{u}_1 + \vec{u}_2$, we have

$A = [\vec{v}_1\ \vec{v}_2] = [2\vec{u}_1\ \ -\vec{u}_1 + \vec{u}_2] = QR$,

where $Q = [\vec{u}_1\ \vec{u}_2] = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$.

III. Clearly, $\det A = -1$, $\det B = 1$, $\det C = -1$.

$\det((3A-4C)^{-1}B^3(3A-4C)^TB^{-1}) = (\det(3A-4C))^{-1}(\det B)^3 \det(3A-4C)(\det B)^{-1} = (\det B)^2 = 1$

$\det\begin{bmatrix} 2A & 0 \\ I_2 - B^T & C^{-1} \end{bmatrix} = \det(2A)\det(C^{-1}) = \frac{4\det A}{\det C} = 4.$

$\det\begin{bmatrix} I_2 & A \\ B & C \end{bmatrix} = \det\begin{bmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & -1 & 1 & \\ 1 & & & -1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & & \\ & & 1 & 1 \\ & & -1 & 1 \\ 1 & -1 & & \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & & \\ 1 & -1 & & \\ & & 1 & 1 \\ & & -1 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\det\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = -4$