Goal: To review important concepts and problems to be covered on Exam 2: Limits, continuity, difference quotients, derivatives, differentiation rules, curve sketching.

PLEASE NOTE that the exam will cover the material from Chapter 2 (Sections 2.1 through 2.10), and the corresponding tutorials: the “Curve sketching and the first and second derivatives” tutorial; the “Derivatives of Trigonometric Functions” tutorial; the “Derivatives, continued” tutorial; and the “Review on limits and continuity” tutorial (which has been reproduced as the first two problems on this review sheet).

1. Define
\[ h(x) = \frac{x^3 - 6x^2 + 11x - 6}{x - 1}. \]

(a) Is \( h(x) \) defined at \( x = 1 \)? Please explain.

(b) Investigate the limit as \( x \to 1 \) of \( h(x) \) numerically, by completing the table below. According to this table, does it appear that this limit exists, and if so, what does this limit appear to be?

<table>
<thead>
<tr>
<th>( x )</th>
<th>.9</th>
<th>.99</th>
<th>.999</th>
<th>.9999</th>
<th>1.0001</th>
<th>1.001</th>
<th>1.01</th>
<th>1.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h(x) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Now note that \( x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6) \). Use this fact to find \( \lim_{x \to 1} h(x) \) algebraically.

(d) Fill in the blanks: At all \( x \) values except \( x = \underline{\text{______}} \), the graph of \( h(x) \) looks like a \underline{\text{______}} that is concave \underline{\text{______}} and has vertex at the point \( \underline{\text{(______,______)}} \). At the point \( x = \underline{\text{______}} \), the graph of \( h(x) \) has a \underline{\text{______}} (reflecting the fact that \( h(x) \) is \underline{\text{______}} at this point).

(e) Now consider the following function:
\[ k(x) = \begin{cases} h(x) & \text{if } x \neq 1, \\ C & \text{if } x = 1, \end{cases} \]

where \( C \) is a constant (and \( h(x) \) is as above). How should we choose this constant \( C \) to make \( k(x) \) continuous at \( x = 1 \)? Please explain.

2. (a) By plugging in some values of \( x \) near zero, investigate
\[ \lim_{x \to 0} \sin \left( \frac{1}{x} \right). \]

Does there appear to be any clear pattern to the behavior of this function as \( x \) approaches zero?
(b) Now suppose you were trying to sketch the graph of \( f(x) = \sin \left( \frac{1}{x} \right) \). You might first try, as we did in class with some other functions, to (a) find the places where \( f(x) = 0 \), and (b) find the places where \( f'(x) = 0 \).

Go ahead and try this, for this function \( f \). In the general vicinity of \( x = 0 \) – let’s say, for \( x \) between \(-1/\pi \) and \( 1/\pi \) – where is \( f(x) = 0 \)? Where is \( f'(x) = 0 \)?

(c) Does your answer to part (b) above give you more insight into what the graph of \( f(x) \) might look like, especially near \( x = 0 \)? If so, try to give a rough sketch of the graph of \( f \) on the axes below.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
\hline
\text{y} & \text{-1} & \text{-1/π} & \text{1/π} & \text{1} \\
\hline
\text{x} & \text{-1/π} & \text{1/π} \\
\hline
\end{array}
\]

(d) Now what do you think about \( \lim_{x \to 0} \sin \left( \frac{1}{x} \right) \)? Explain.

3. Using the limit definition, compute the derivative of the function \( f(x) = \frac{4}{x} \) at the point \( x = 2 \).

4. Give the best possible estimate for \( g'(1) \) that can be obtained from the following table:
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5. Let

\[ g(x) = \begin{cases} 
  x^2 + \cos x & \text{if } -5 < x < 0, \\
  5 & \text{if } x = 0, \\
  e^{3x/2} & \text{if } 0 < x < 5.
\]

Answer the following questions about the function \( g(x) \).

(a) \( \lim_{x \to 2^+} g(x) = \)

(b) \( \lim_{x \to 0^+} g(x) = \)

(c) \( \lim_{x \to 0^-} g(x) = \)

(d) \( \lim_{x \to 0} g(x) = \)

6. Sketch, on the axes below, one possible function \( f(x) \) that satisfies ALL of the following properties.

(a) \( \lim_{x \to 1^+} f(x) = 2 \)

(b) \( \lim_{x \to 1} f(x) \) does not exist.

(c) \( \lim_{x \to \infty} f(x) = -4 \)

(d) \( \lim_{x \to -4^+} f(x) = -\infty \)

(e) \( f(x) \) is continuous, increasing and concave up on \( (-\infty, -4) \).

<table>
<thead>
<tr>
<th>x</th>
<th>g(x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.98</td>
<td>0.8647</td>
</tr>
<tr>
<td>.99</td>
<td>0.9312</td>
</tr>
<tr>
<td>1</td>
<td>1.0000</td>
</tr>
<tr>
<td>1.01</td>
<td>1.0712</td>
</tr>
<tr>
<td>1.02</td>
<td>1.1449</td>
</tr>
</tbody>
</table>
7. Explain the relationship between continuity and limits.

8. Explain the relationship between the derivative and limits.

9. Explain the relationship between continuity and differentiability.

10. Using the following table, determine the value of $x$ (from those given) where $f'(x)$ is closest to 6. (Assume the function is well behaved, i.e. the set of points below gives a good idea to the general shape of the function.)

$$
\begin{array}{|c||c|c|c|c|c|c|c|c|}
\hline
x & 2.0 & 2.2 & 2.4 & 2.6 & 2.8 & 3.0 & 3.2 & 3.4 & 3.6 \\
\hline
\hline
\end{array}
$$

11. Let

$$
f(x) = 3 - |x| = \begin{cases} 
3 - x & \text{if } x \geq 0, \\
3 + x & \text{if } x < 0.
\end{cases}
$$

(a) In the space provided, draw a graph of the function $f(x)$.
(b) Using your graph from part (a), and your understanding of the derivative as the rate of change/slope of the tangent line, find the derivative function \( f'(x) \) of the above function \( f(x) \), for \( x \) not equal to 0 (fill in the blanks):

\[
f'(x) = \begin{cases} 
\text{__________} & \text{if } x > 0, \\
\text{__________} & \text{if } x < 0.
\end{cases}
\]

(c) But what about \( f'(0) \)? The slope of the line tangent to the graph of \( f(x) \) at \( x = 0 \) is not so clear from the picture.

So let’s try to compute

\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}
\]

by first looking at the corresponding lefthand and righthand limits.

(i) Compute \( \lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} \). Hint: use the “piecewise” definition of \( f(x) \) given above.

(ii) Compute \( \lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} \).

(iii) What do your answers to parts (i) and (ii) tell you about \( f'(0) \)? Please explain.

12. Let

\[ f(x) = \ln(1 + \sin x). \]

(a) What is the domain of \( f(x) \)?

(b) Compute \( f'(x) \) and \( f''(x) \).

(c) Show that \( f(x) \) satisfies the differential equation

\[ e^{-f(x)} + f''(x) = 0. \]

13. Let

\[ g(x) = \frac{x}{1 + x^2}. \]

Sketch the graph of \( g(x) \), on the axes below. Be sure to label all local maxima and minima, and inflection points, on your graph.
14. Evaluate each of the following derivatives. Please show all of your work.

(a) \( \frac{d}{dx}[x^2 \sin 3x] \)  
(b) \( \frac{d}{dx}[(\sin(3x))^2] \)  
(c) \( \frac{d}{dx}[xe^{\cos(2x)}] \)  
(d) \( \frac{d}{dx}[e^{x+\cos(2x)}] \)  
(e) \( \frac{d}{dx}[e^{x\sin(3x)}] \)  
(f) \( \frac{d}{dx}\left[ \frac{x}{e^{\sin(3x)}} \right] \)  
(g) \( \frac{d}{dx}[e^{3x}\cos(e^x)] \)  
(h) \( \frac{d}{dx}[\cos(x + e^{3x})] \)

15. On the axes below is a graph of the function \( f(x) = \sin x \), for \(-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\).

(a) Explain why this function \( f(x) \), when restricted to the given domain, has an inverse function \( f^{-1} \).
(b) Sketch the graph of $f^{-1}$, on the axes below.

(c) Find $\frac{d}{dx}[f^{-1}(x)]$. Hint: to express your final answer in a form that does not explicitly involve $f^{-1}(x)$, it might be useful to note that, for any number $\theta$ between $-\pi/2$ and $\pi/2$, $\cos \theta = \sqrt{1 - (\sin \theta)^2}$. 