**Goal:** To better understand the asymptotes of a rational function and, in particular, to explore the role of long division in understanding those asymptotes when they are not horizontal.

In this worksheet we examine rational functions, that is functions of the form \( f(x) = \frac{P(x)}{Q(x)} \)
where \( P(x) \) and \( Q(x) \) are polynomials. Recall that the degree of a polynomial is the highest power of \( x \) appearing in the polynomial, e.g, the degree of the polynomial \( P(x) = 2x^4 - 3x + 2 \) is 4 while the degree of \( Q(x) = x - 3 \) is 1.

In this worksheet we will discover that the non-vertical asymptotes of a rational function will depend, in part, on the relationship between the degree of its numerator and the degree of its denominator.

**IMPORTANT ASSUMPTION** Henceforth we will only consider rational functions of the form \( f(x) = \frac{P(x)}{Q(x)} \) where the fraction \( \frac{P(x)}{Q(x)} \) cannot be simplified, in other words where the polynomials \( P(x) \) and \( Q(x) \) do not have a common factor.

**Preliminary question 1.** If you are presented with a rational function \( f(x) = \frac{P(x)}{Q(x)} \) which can be simplified, why do you think you should simplify it before you try to understand it’s graph?
By canceling as many common factors as possible between \( P(x) \) and \( Q(x) \) we will have an expression that is, possibly, easier to work with. It is very important to use the original form of the function to determine its domain, but the simplified form may be used to locate vertical asymptotes.

**Preliminary question 2.** What, in your own words, is an asymptote? (After you complete each of the problems below compare what you have discovered with your answer to this question.)
One possible answer: A asymptote is a line the graph of the function gets closer and closer to but does not touch.
Case 1. \( f(x) = \frac{P(x)}{Q(x)} \) where the degree of \( Q(x) \) is greater than the degree of \( P(x) \).

An example. Let \( f(x) = \frac{-x^2 - x + 7}{2x^4 - 3x + 2} \)

(a) Evaluate each of the limits: \( \lim_{x \to \infty} \frac{-x^2 - x + 7}{2x^4 - 3x + 2} \) and \( \lim_{x \to -\infty} \frac{-x^2 - x + 7}{2x^4 - 3x + 2} \).

\[
\lim_{x \to \infty} \frac{-x^2 - x + 7}{2x^4 - 3x + 2} = \lim_{x \to \infty} \frac{1 + 1/x - 7/x^2}{2x^2 + 3/x - 2/x^2} = 0
\]

and

\[
\lim_{x \to -\infty} \frac{-x^2 - x + 7}{2x^4 - 3x + 2} = \lim_{x \to -\infty} \frac{1 + 1/x - 7/x^2}{2x^2 + 3/x - 2/x^2} = 0
\]

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

\[
f(x) = \frac{-x^2 - x + 7}{2x^4 - 3x + 2}
\]

The \( y \)-coordinates of points on the graph become arbitrarily close to 0 as \( |x| \to \infty \). So the line \( y = 0 \), which is the \( x \)-axis, is an asymptote.

Conclusion. Do you think you will obtain exactly the same horizontal asymptote for any rational function \( f(x) = \frac{P(x)}{Q(x)} \) where the degree of \( Q(x) \) is greater than the degree of \( P(x) \)? (Explain your answer.)

Yes. Dividing the numerator and denominator by \( x^{\text{degree of } P(x)} \), i.e., by the highest power of \( x \) in the numerator, will always give a ratio whose numerator approaches some finite, nonzero value, and whose denominator still contains a positive power of \( x \).
**Case 2.** $f(x) = \frac{P(x)}{Q(x)}$ where the degree of $Q(x)$ equals the degree of $P(x)$.

(a) Evaluate each of the limits: $\lim_{x \to \infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$ and $\lim_{x \to -\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$.

$$
\lim_{x \to \infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2} = \lim_{x \to \infty} \frac{-1 - 1/x^2 + 7/x^3}{2 - 3/x^2 + 2/x^3} = \frac{-1}{2}
$$

and

$$
\lim_{x \to -\infty} \frac{-x^3 - x + 7}{2x^3 - 3x + 2} = \lim_{x \to -\infty} \frac{-1 - 1/x^2 + 7/x^3}{2 - 3/x^2 + 2/x^3} = \frac{-1}{2}
$$

(b) What do each of the limits in part (a) tell you about the non-vertical asymptotes of the function

$$f(x) = \frac{-x^3 - x + 7}{2x^3 - 3x + 2}$$

The line $y = -1/2$ is a horizontal asymptote.

**Conclusion.** Do you think a rational function $f(x) = \frac{P(x)}{Q(x)}$, where the degree of $Q(x)$ equals the degree of $P(x)$, will always have a horizontal asymptote? (Explain your answer.)

Yes. Dividing the numerator and denominator by the $x$-degree of $P(x)$, i.e. by the highest power of $x$ in the numerator (which in this case is also the highest power of $x$ appearing in the denominator), will always give a ratio whose numerator approaches some finite, nonzero value, and whose denominator also approaches some finite, nonzero value. The asymptote will then be the line

$$y = \frac{\text{coefficient of the highest power of } x \text{ in the numerator}}{\text{coefficient of the highest power of } x \text{ in the denominator}}.$$
Case 3. \( f(x) = \frac{P(x)}{Q(x)} \) where the degree of \( Q(x) \) is 1 less than the degree of \( P(x) \).

(a) Evaluate each of the limits: \( \lim_{x \to \infty} \frac{x^3 - x}{2x^2 + 2} \) and \( \lim_{x \to -\infty} \frac{x^3 - x}{2x^2 + 2} \).

\[
\lim_{x \to \infty} \frac{x^3 - x}{2x^2 + 2} = \lim_{x \to \infty} \frac{x - \frac{1}{x}}{2 + \frac{2}{x^2}} = \infty
\]

and

\[
\lim_{x \to -\infty} \frac{x^3 - x}{2x^2 + 2} = \lim_{x \to -\infty} \frac{x - \frac{1}{x}}{2 + \frac{2}{x^2}} = -\infty
\]

(b) What, if anything, do each of the limits in part (a) tell you about possible non-vertical asymptotes of the function

\[ f(x) = \frac{x^3 - x}{2x^2 + 2}. \]

They tell us that the function does not have any horizontal asymptotes.

How to discover the asymptotes in the above problem. If we divide the numerator into the denominator of \( f(x) = \frac{x^3 - x}{2x^2 + 2} \) we discover that

\[ f(x) = \frac{x^3 - x}{2x^2 + 2} = \frac{1}{2} x + \frac{-2x}{2x^2 + 2}. \]

(c) Calculate each of the limits: \( \lim_{x \to \infty} \frac{-2x}{2x^2 + 2} \) and \( \lim_{x \to -\infty} \frac{-2x}{2x^2 + 2} \).

Since the power of \( x \) in the denominator is 1 more than the power of \( x \) in the numerator, each of these limits equals 0 (see Case 1).
(d) Can you explain how your results in (c) allow you to conclude that for $x$ with $|x|$ very large

$$f(x) = \frac{x^3 - x}{2x^2 + 2} \approx \frac{1}{2}x.$$ 

Since

$$f(x) = \frac{x^3 - x}{2x^2 + 2} = \frac{1}{2}x + \frac{-2x}{2x^2 + 2} \quad \text{and} \quad \frac{-2x}{2x^2 + 2} \to 0 \quad \text{as} \quad x \to \infty \quad \text{and} \quad x \to -\infty,$$

we know that the expression $\frac{x^3 - x}{2x^2 + 2} \to \frac{1}{2}x$ as $x \to \infty$ and $x \to -\infty$. This means that for $x$ with $|x|$ very large

$$f(x) = \frac{x^3 - x}{2x^2 + 2} \approx \frac{1}{2}x.$$

**Conclusion.** In the example above, the graph of $f(x)$ is becoming arbitrarily close to the line $y = \frac{1}{2}x$, in other words that line is an asymptote for the function! Do you think a rational function $f(x) = \frac{P(x)}{Q(x)}$, where the degree of $Q(x)$ is 1 less than the degree of $P(x)$, will always have a line as an asymptote? (Explain your answer.)

Yes, and here’s why. We can divide $Q(x)$ into $P(x)$ using long division of polynomials, to get a quotient of the form $ax + b$ (it’s of this form because the degree of $P(x)$ is one larger than the degree of $Q(x)$), and a remainder $R(x)$ whose degree is smaller than that of $Q(x)$ (it’s smaller because that’s what it means for $R(x)$ to be the remainder). In other words, we have

$$\frac{P(x)}{Q(x)} = ax + b + \frac{R(x)}{Q(x)},$$

where $\lim_{x \to \infty} \frac{R(x)}{Q(x)} = 0$ (because $Q(x)$ has a higher power of $x$ in it than does $R(x)$). Then the asymptote is $y = ax + b$.

**Final question.** In tonight’s homework you will discover that a rational function can have a curved asymptote. How will this discovery force you to reconsider your original definition of an asymptote above? We discovered in this worksheet an asymptote need not be horizontal nor vertical. Now you tell me an asymptote can be curved? How cool is that (and my original definition was so way off).