Name: ________________________________  Math 1300

Worksheet: Derivatives of inverse trig functions: Answers in Red

Goal: To find the derivatives of the inverse trig functions.

1. Explain why the function \( f(x) = \sin(x) \) does not have an inverse function unless we restrict its domain.

   Solution. In order to have an inverse function over some interval the function must be 1-to-1 on that interval (graphically the function must satisfy the horizontal line test).

2. Explain why the function \( f(x) = \sin(x) \) does have an inverse function if we restrict its domain to the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\). (The sine function with this restricted domain is sometimes denoted by Sin(x).)

   Solution. The sine function is not 1-to-1 on its entire domain (with is the entire real line) but it is 1-to-1 on the interval \([-\frac{\pi}{2}, \frac{\pi}{2}]\).

3. Relationships involving trig and inverse trig functions can sometimes be simplified using elementary triangle trigonometry. For example, draw a right triangle below whose hypotenuse equals 1 and one of whose non-right angles is denoted by \( \theta \). If we let \( \theta = \sin^{-1}(x) = \sin^{-1}(\frac{x}{1}) \), with \( 0 < x < 1 \), then \( x \) is one of the sides of the triangle. Figure out which side and label it on your drawing.

   ![Diagram of a right triangle](image)

   Use the above triangle to express \( \cos(\sin^{-1}(x)) \) in terms of \( x \).

   Solution. Since \( \sin^{-1}(x) \) is the angle \( \theta \), finding \( \cos(\sin^{-1}(x)) \) is the same as finding \( \cos(\theta) \). Recall that in a right triangle \( \cos(\theta) = \frac{\text{the opposite side}}{\text{the hypotenuse}} \) so

   \[
   \cos(\sin^{-1}(x)) = \cos(\theta) = \frac{\sqrt{1 - x^2}}{1} = \sqrt{1 - x^2}.
   \]
4. Differentiate the relationship

\[ \sin(\sin^{-1}(x)) = x \]

using the chain rule, to find a formula for \( \frac{d}{dx} \sin^{-1}(x) \) in terms of trigonometric and inverse trigonometric functions. Then use the relationship you found in problem 3 to express \( \frac{d}{dx} \sin^{-1}(x) \) without using any trig functions. **Solution.**

From \( \sin(\sin^{-1}(x)) = x \) we can conclude that \( \frac{d}{dx} \sin(\sin^{-1}(x)) = \frac{d}{dx} x \)

We apply the chain rule to the left hand side of this last equation and obtain:

\[ \cos(\sin^{-1}(x)) \times \frac{d}{dx} \sin^{-1}(x) = 1 \]

and therefore

\[ \frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}. \]

5. We know that the domain of \( \sin^{-1}(x) \) is the interval \((-1, 1)\). In your drawing of the triangle in Problem 3 you probably assumed that \( x \) was positive, so \( 0 < x < 1 \). Draw a picture, similar to the one in Problem 3 above, that works for negative values of \( x \) (that is, for \( x \) with \(-1 < x < 0\)), and use the chain rule and this picture to find \( \frac{d}{dx} \sin^{-1}(x) \) for such \( x \).

**Solution.**

Even when \( x \) is the negative, as pictured above, we have \( \sin(\sin^{-1}(x)) = x \), so the deduction in Problem 4 is still valid.
6. The function \( f(x) = \tan(x) \) has an inverse function, call it \( \tan^{-1}(x) \), if we restrict its domain to the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\). Find the derivative of \( \tan^{-1}(x) \) as follows: first, differentiate the equation

\[
\tan(\tan^{-1}(x)) = x
\]

using the chain rule, to find a formula for \( \frac{d}{dx} \tan^{-1}(x) \) in terms of trigonometric and inverse trigonometric functions. Then, use a right triangle similar to the one in Problem 3 above to express \( \frac{d}{dx} \tan^{-1}(x) \) without using any trig functions. (Be careful to describe the domain and range of \( \tan^{-1}(x) \).)

Solution. \( f(x) = \tan(x) \) is 1-to-1 on the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) (why did we use an open interval for \( \tan(x) \) and a closed interval for \( \sin(x) \)?) So we differentiate the basic relationship \( \tan(\tan^{-1}(x)) = x \) with respect to \( x \), using the chain rule on the left hand side, and obtain:

\[
\sec^2(\tan^{-1}(x)) \times \frac{d}{dx} \tan^{-1}(x) = 1
\]

and therefore

\[
\frac{d}{dx} \tan^{-1}(x) = \frac{1}{\sec^2(\tan^{-1}(x))}.
\]

For \( x > 0 \) we can use the triangle,

\[
\sqrt{1 + x^2}
\]

where \( \tan^{-1}(x) = \theta \) to find that

\[
\sec^2(\tan^{-1}(x)) = \sec^2(\theta) = 1 + x^2.
\]

Thus we have discovered that for \( x \) positive,

\[
\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1 + x^2}.
\]

The same argument works for \( x < 0 \) as in Problem 5.