1.2 The spread of disease: the *SIR* model

Many human diseases are contagious: you "catch" them from someone who is already infected. Contagious diseases are of many kinds. Smallpox, polio, plague, and Ebola are severe and even fatal, while the common cold and the childhood illnesses of measles, mumps, and rubella are usually relatively mild. Moreover, you can catch a cold over and over again, but you get measles only once. A disease like measles is said to "confer immunity" on someone who recovers from it.

Some diseases have the potential to affect large segments of a population; they are called *epidemics* (from the Greek words *epi*, upon + *demos*, the people.) *Epidemiology* is the scientific study of these diseases.

An epidemic is a complicated matter, but the dangers posed by contagion – and especially by the appearance of new and uncontrollable diseases – compel us to learn as much as we can about the nature of epidemics. Mathematics offers a very special kind of help.

First, we can try to draw out of the situation its essential features and describe them mathematically. This is calculus as *language*. We substitute an "ideal" mathematical world for the real one. This mathematical world is called a **model**. Second, we can use mathematical insights and methods to analyze the model. This is calculus as *tool*. Any conclusion we reach about the model can then be interpreted to tell us something about the reality.

To give you an idea how this process works, we'll build a model – called the SIR model, for *susceptible, infected, recovered* – of an epidemic. This is a well-known model in epidemiology. Its basic purpose is to help us understand the way a contagious disease spreads through a population – to the point where we can even predict what fraction will fall ill, and when. Let's suppose the disease we want to model is like measles.

Initial setup

Our disease will entail the following three quantities, and their rates of change (indicated by primes, as described in Section 1.1).

S: number of susceptible individuals	S': rate of change of S
I: number of infected individuals	I': rate of change of I
R: number of recovered individuals	R': rate of change of R

We make the following assumptions about the nature of this disease:

- Everyone recovers eventually.
- The duration of infection is the same for everyone.
- Once recovered, you're immune, and can no longer infect anyone.
- Only a fraction of contacts with the disease cause infection.

- The units of S, I, and R are persons.
- The units of time are days.
- The units of S', I', and R' are persons per day, written person/day.
- The system is *closed;* this simply means that the total size of the population, which equals the sum $S + I + R$, does not change.

Thinking about S', I', and R'

In Example 1.1.5 above, we used some basic assumptions about the natures of the infected and recovered populations to develop a rate equation for the latter. This rate equation, equation (1.1.3) above, is *autonomous* – this means that R' is expressed explicitly in terms of R itself.

In this section, we will develop a *system* of rate equations – one for each of the three variables S, I, and R. This system will also be autonomous, in the sense that the rates of change of the quantities in question will be expressed in terms of the quantities themselves. In later sections, we'll use *Euler's method*, described at the end of the previous section, to chart the evolution of our epidemic.

Let's begin by addressing R'. (We deal with this rate of change first because its analysis is, in many ways, the easiest of the three.)

Suppose infection lasts for k days. Also assume, in the absence of any definite information to the contrary, that the infected population is "uniform with respect to duration of infection," at any given point in time. That is, assume that there are, any any instant, just as many people in this population who have been infected for one day as there are who have been infected for two, or three, and so on, up to k days.

Then on any given day, one kth of the infected population will recover. In other words, the *rate of recovery*, in individuals per day, is equal to $1/k$ times I. Recall that we are calling this rate R'. So, in symbols:

$$\boxed{\boldsymbol{R' = bI, \quad \text{where} \quad b = \frac{1}{k}.}}$$

Rate equation for the recovered population

Here, b is constant, in that it doesn't change over the course of time. Of course, different diseases may entail different values of b. In mathematical modeling, a number that is constant within a given situation, but may vary from situation to situation, is called a *parameter*. Often, we will use uppercase letters for the variables (other than the time variable) in rate equations, and will use lowercase letters for the parameters (and the time variable).

When one variable quantity equals a constant times another, we say the two quantities are *proportional*. So in our present model, the *rate of change* of the recovered population is proportional to the *size* of the infected population.

Let's move on to examination of S', which is the next easiest rate of change to model. Let's suppose that:

(i) Each susceptible individual comes into contact with a proportion, call it p, of the infected population each day. This implies that each susceptible person has contact with pI infected persons per day. This in turn implies that there are $pI \times S = pSI$ *total* contacts between susceptible and infected individuals each day.

(ii) A certain proportion, call it q, of the above contacts cause infection.

The above tells us that there are $q \times pSI$ new infections occurring each day. Which in turn implies that the size of the susceptible population *decreases* by $qpSI$ persons each day. In other words, in persons/day, we have

$$\boxed{\boldsymbol{S' = -aSI, \quad \text{where} \quad a = qp.}}$$

Rate equation for the susceptible population

The minus sign denotes a *negative* rate of change, meaning a *decrease* in the quantity in question.

Finally, we consider I'. Since $S + I + R$ is assumed to be constant, the sum $S' + I' + R'$ of the rates of change of the three subpopulations must be *zero* – any change in one of these quantities is offset by changes in the others. That is, by the above formulas for S' and R',

$$0 = S' + I' + R' = -aSI + I' + bI$$

or, solving for I',

$$\boxed{\boldsymbol{I' = aSI - bI.}}$$

Rate equation for infected population

To summarize: under the conditions described above, we have

$$\boxed{\begin{aligned} \boldsymbol{S'} &= \boldsymbol{-aSI} \\ \boldsymbol{I'} &= \boldsymbol{aSI - bI} \\ \boldsymbol{R'} &= \boldsymbol{\phantom{-aSI - {}} bI} \end{aligned}}$$

The *SIR* equations

Here:

- The number a is a positive parameter called the *transmission coefficient*. Recall from above that $a = qp$, where p is the proportion, or fraction, of the infected population with which each infected comes into contact each day, and q is the proportion these of contacts that cause infection. Note that the *units* of a are $1/$(person-day), or "inverse person-days." Why? Because the units on both sides of any equation must agree. So, considering for example the equation $S' = -aSI$, we see that inverse person-days are indeed the correct units for a, because they insure that the right-hand side of this equation has units

$$\frac{1}{\text{person-day}} \times \text{persons} \times \text{persons} = \frac{\text{persons}}{\text{day}},$$

  which are the units of the left-hand side of this equation.

- The number b is a positive parameter called the *recovery coefficient*. Recall from above that $b = 1/k$, where k is the number of days that infection lasts. The units of b are $1/$day, or inverse days.

As noted above, S is a *decreasing* quantity. This is reflected in the above equation for S': the quantities a, R, and I are all positive, so $S' = -aSI$ is negative. And again, a negative rate of change connotes a decreasing quantity. Actually, it's conceivable that I or R might be zero at some point(s), in which case S' would be zero there too. So technically, it might be more precise to say that S is *nonincreasing*: its rate of change is never positive. To keep the terminology simple, though, we'll use the term "decreasing" even for quantities that are, strictly speaking, nonincreasing.

Similarly, R is an increasing quantity. The size I of the infected polulation can increase *or* decrease, since the sign of $I' = aSI - bI$ can be positive or negative, depending on the relative sizes of aSI and bI. In the next section, we'll look at the rise and fall of I more closely.

Also in the next section, we'll use our SIR equations, together with a generalized version of the *Euler's method* discussed in the previous section, to study the progress of our epidemic.