DISTANCE, CIRCLES, AND QUADRATIC EQUATIONS

DISTANCE BETWEEN TWO POINTS IN THE PLANE

Suppose that we are interested in finding the distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the xy-plane. If, as in Figure G.1, we form a right triangle with P_1 and P_2 as vertices, then it follows from Theorem B.4 in Appendix B that the sides of that triangle have lengths $|x_2 - x_1|$ and $|y_2 - y_1|$. Thus, it follows from the Theorem of Pythagoras that

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and hence we have the following result.

G.1 THEOREM. *The distance d between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in a coordinate plane is given by*

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

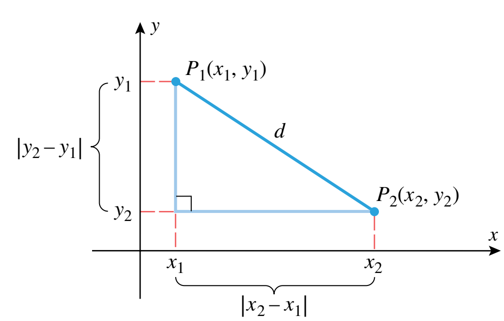


Figure G.1

To apply Formula (1) the scales on the coordinate axes must be the same; otherwise, we would not have been able to use the Theorem of Pythagoras in the derivation. Moreover, when using Formula (1) it does not matter which point is labeled P_1 and which one is labeled P_2, since reversing the points changes the signs of $x_2 - x_1$ and $y_2 - y_1$; this has no effect on the value of d because these quantities are squared in the formula. When it is important to emphasize the points, the distance between P_1 and P_2 is denoted by $d(P_1, P_2)$ or $d(P_2, P_1)$.

Example 1 Find the distance between the points $(-2, 3)$ and $(1, 7)$.

Solution. If we let (x_1, y_1) be $(-2, 3)$ and let (x_2, y_2) be $(1, 7)$, then (1) yields

$$d = \sqrt{[1 - (-2)]^2 + [7 - 3]^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$ ◀

▶ **Example 2** It can be shown that the converse of the Theorem of Pythagoras is true; that is, if the sides of a triangle satisfy the relationship $a^2 + b^2 = c^2$, then the triangle must be a right triangle. Use this result to show that the points $A(4, 6)$, $B(1, -3)$, and $C(7, 5)$ are vertices of a right triangle.

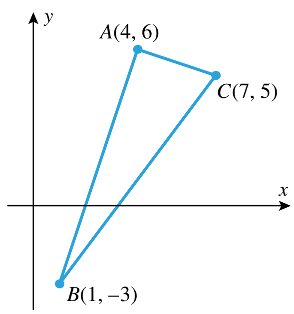


Figure G.2

Solution. The points and the triangle are shown in Figure G.2. From (1), the lengths of the sides of the triangles are

$$d(A, B) = \sqrt{(1-4)^2 + (-3-6)^2} = \sqrt{9+81} = \sqrt{90}$$
$$d(A, C) = \sqrt{(7-4)^2 + (5-6)^2} = \sqrt{9+1} = \sqrt{10}$$
$$d(B, C) = \sqrt{(7-1)^2 + [5-(-3)]^2} = \sqrt{36+64} = \sqrt{100} = 10$$

Since

$$[d(A, B)]^2 + [d(A, C)]^2 = [d(B, C)]^2$$

it follows that ΔABC is a right triangle with hypotenuse BC. ◀

THE MIDPOINT FORMULA

It is often necessary to find the coordinates of the midpoint of a line segment joining two points in the plane. To derive the midpoint formula, we will start with two points on a coordinate line. If we assume that the points have coordinates a and b and that $a \le b$, then, as shown in Figure G.3, the distance between a and b is $b - a$, and the coordinate of the midpoint between a and b is

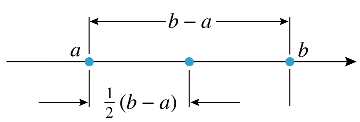


Figure G.3

$$a + \tfrac{1}{2}(b-a) = \tfrac{1}{2}a + \tfrac{1}{2}b = \tfrac{1}{2}(a+b)$$

which is the arithmetic average of a and b. Had the points been labeled with $b \le a$, the same formula would have resulted (verify). Therefore, *the midpoint of two points on a coordinate line is the arithmetic average of their coordinates, regardless of their relative positions.*

If we now let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points in the plane and $M(x, y)$ the midpoint of the line segment joining them (Figure G.4), then it can be shown using similar triangles that x is the midpoint of x_1 and x_2 on the x-axis and y is the midpoint of y_1 and y_2 on the y-axis, so

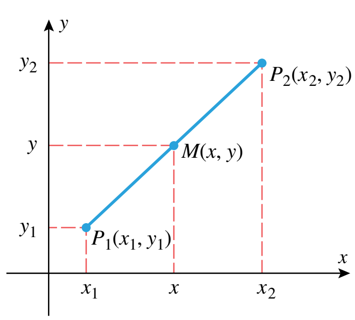


Figure G.4

$$x = \tfrac{1}{2}(x_1 + x_2) \quad \text{and} \quad y = \tfrac{1}{2}(y_1 + y_2)$$

Thus, we have the following result.

> **G.2 THEOREM (*The Midpoint Formula*).** *The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) in a coordinate plane is*
>
> $$\left(\tfrac{1}{2}(x_1 + x_2), \tfrac{1}{2}(y_1 + y_2)\right) \tag{2}$$

▶ **Example 3** Find the midpoint of the line segment joining $(3, -4)$ and $(7, 2)$.

Solution. From (2) the midpoint is

$$\left(\tfrac{1}{2}(3+7), \tfrac{1}{2}(-4+2)\right) = (5, -1) \quad ◀$$

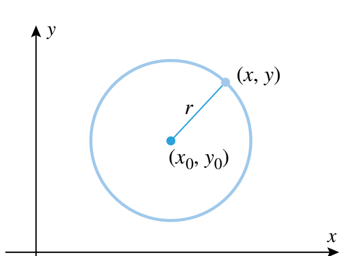


Figure G.5

CIRCLES

If (x_0, y_0) is a fixed point in the plane, then the circle of radius r centered at (x_0, y_0) is the set of all points in the plane whose distance from (x_0, y_0) is r (Figure G.5). Thus, a point

(x, y) will lie on this circle if and only if

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$$

or equivalently,

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \tag{3}$$

This is called the ***standard form of the equation of a circle***.

▶ Example 4 Find an equation for the circle of radius 4 centered at $(-5, 3)$.

Solution. From (3) with $x_0 = -5$, $y_0 = 3$, and $r = 4$ we obtain

$$(x + 5)^2 + (y - 3)^2 = 16$$

If desired, this equation can be written in an expanded form by squaring the terms and then simplifying:

$$(x^2 + 10x + 25) + (y^2 - 6y + 9) - 16 = 0$$

$$x^2 + y^2 + 10x - 6y + 18 = 0$$ ◀

▶ Example 5 Find an equation for the circle with center $(1, -2)$ that passes through $(4, 2)$.

Solution. The radius r of the circle is the distance between $(4, 2)$ and $(1, -2)$, so

$$r = \sqrt{(1 - 4)^2 + (-2 - 2)^2} = 5$$

We now know the center and radius, so we can use (3) to obtain the equation

$$(x - 1)^2 + (y + 2)^2 = 25 \quad \text{or} \quad x^2 + y^2 - 2x + 4y - 20 = 0$$ ◀

FINDING THE CENTER AND RADIUS OF A CIRCLE

When you encounter an equation of form (3), you will know immediately that its graph is a circle; its center and radius can then be found from the constants that appear in the equation:

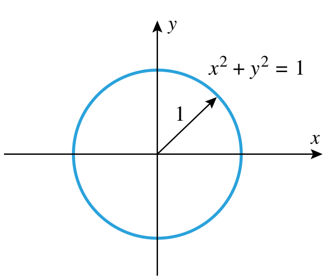
$$(x - x_0)^2 \quad + \quad (y - y_0)^2 \quad = \quad r^2$$

x-coordinate of the center is x_0 | y-coordinate of the center is y_0 | radius squared

▶ Example 6

EQUATION OF A CIRCLE	CENTER (x_0, y_0)	RADIUS r
$(x - 2)^2 + (y - 5)^2 = 9$	$(2, 5)$	3
$(x + 7)^2 + (y + 1)^2 = 16$	$(-7, -1)$	4
$x^2 + y^2 = 25$	$(0, 0)$	5
$(x - 4)^2 + y^2 = 5$	$(4, 0)$	$\sqrt{5}$

◀



The unit circle

Figure G.6

The circle $x^2 + y^2 = 1$, which is centered at the origin and has radius 1, is of special importance; it is called the ***unit circle*** (Figure G.6).

OTHER FORMS FOR THE EQUATION OF A CIRCLE

An alternative version of Equation (3) can be obtained by squaring the terms and simplifying. This yields an equation of the form

$$x^2 + y^2 + dx + ey + f = 0 \tag{4}$$

where d, e, and f are constants. (See the final equations in Examples 4 and 5.)

Still another version of the equation of a circle can be obtained by multiplying both sides of (4) by a nonzero constant A. This yields an equation of the form

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \tag{5}$$

where A, D, E, and F are constants and $A \neq 0$.

If the equation of a circle is given by (4) or (5), then the center and radius can be found by first rewriting the equation in standard form, then reading off the center and radius from that equation. The following example shows how to do this using the technique of ***completing the square***. In preparation for the example, recall that completing the square is a method for rewriting an expression of the form

$$x^2 + bx$$

as a difference of two squares. The procedure is to take half the coefficient of x, square it, and then add and subtract that result from the original expression to obtain

$$x^2 + bx = x^2 + bx + (b/2)^2 - (b/2)^2 = [x + (b/2)]^2 - (b/2)^2$$

▶ Example 7 Find the center and radius of the circle with equation

(a) $x^2 + y^2 - 8x + 2y + 8 = 0$ (b) $2x^2 + 2y^2 + 24x - 81 = 0$

Solution (a). First, group the x-terms, group the y-terms, and take the constant to the right side:

$$(x^2 - 8x) + (y^2 + 2y) = -8$$

Next we want to add the appropriate constant within each set of parentheses to complete the square, and subtract the same constant outside the parentheses to maintain equality. The appropriate constant is obtained by taking half the coefficient of the first-degree term and squaring it. This yields

$$(x^2 - 8x + 16) - 16 + (y^2 + 2y + 1) - 1 = -8$$

from which we obtain

$$(x - 4)^2 + (y + 1)^2 = -8 + 16 + 1 \qquad \text{or} \qquad (x - 4)^2 + (y + 1)^2 = 9$$

Thus from (3), the circle has center $(4, -1)$ and radius 3.

Solution (b). The given equation is of form (5) with $A = 2$. We will first divide through by 2 (the coefficient of the squared terms) to reduce the equation to form (4). Then we will proceed as in part (a) of this example. The computations are as follows:

$$x^2 + y^2 + 12x - \tfrac{81}{2} = 0 \qquad \text{We divided through by 2.}$$

$$(x^2 + 12x) + y^2 = \tfrac{81}{2}$$

$$(x^2 + 12x + 36) + y^2 = \tfrac{81}{2} + 36 \qquad \text{We completed the square.}$$

$$(x + 6)^2 + y^2 = \tfrac{153}{2}$$

From (3), the circle has center $(-6, 0)$ and radius $\sqrt{\frac{153}{2}}$. ◀

DEGENERATE CASES OF A CIRCLE

There is no guarantee that an equation of form (5) represents a circle. For example, suppose that we divide both sides of (5) by A, then complete the squares to obtain

$$(x - x_0)^2 + (y - y_0)^2 = k$$

Depending on the value of k, the following situations occur:

- $(k > 0)$ The graph is a circle with center (x_0, y_0) and radius $\sqrt{k}$.
- $(k = 0)$ The only solution of the equation is $x = x_0$, $y = y_0$, so the graph is the single point (x_0, y_0).
- $(k < 0)$ The equation has no real solutions and consequently no graph.

Example 8 Describe the graphs of

(a) $(x-1)^2 + (y+4)^2 = -9$ (b) $(x-1)^2 + (y+4)^2 = 0$

Solution (a). There are no real values of x and y that will make the left side of the equation negative. Thus, the solution set of the equation is empty, and the equation has no graph.

Solution (b). The only values of x and y that will make the left side of the equation 0 are $x = 1$, $y = -4$. Thus, the graph of the equation is the single point $(1, -4)$. ◀

The following theorem summarizes our observations.

The last two cases in Theorem G.3 are called ***degenerate cases***. In spite of the fact that these degenerate cases can occur, (6) is often called the ***general equation of a circle***.

G.3 THEOREM. *An equation of the form*

$$Ax^2 + Ay^2 + Dx + Ey + F = 0 \tag{6}$$

where $A \neq 0$, *represents a circle, or a point, or else has no graph.*

THE GRAPH OF $y = ax^2 + bx + c$

An equation of the form

$$y = ax^2 + bx + c \quad (a \neq 0) \tag{7}$$

is called a ***quadratic equation in x***. Depending on whether a is positive or negative, the graph, which is called a ***parabola***, has one of the two forms shown in Figure G.7. In both cases the parabola is symmetric about a vertical line parallel to the y-axis. This line of symmetry cuts the parabola at a point called the ***vertex***. The vertex is the low point on the curve if $a > 0$ and the high point if $a < 0$.

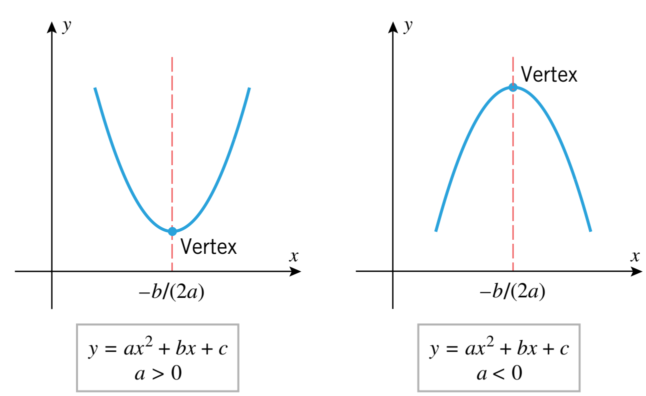


Figure G.7

In the exercises (Exercise 78) we will help the reader show that the x-coordinate of the vertex is given by the formula

$$x = -\frac{b}{2a} \tag{8}$$

x	$y = x^2 - 2x - 2$
−1	1
0	−2
1	−3
2	−2
3	1

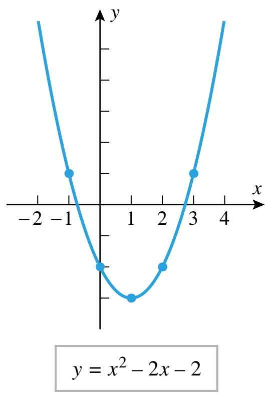


Figure G.8

x	$y = -x^2 + 4x - 5$
0	−5
1	−2
2	−1
3	−2
4	−5

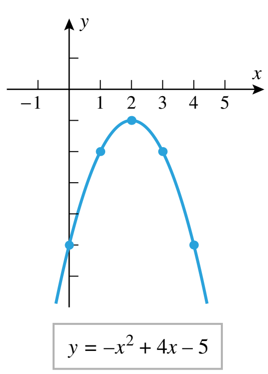


Figure G.9

With the aid of this formula, a reasonably accurate graph of a quadratic equation in x can be obtained by plotting the vertex and two points on each side of it.

▶ **Example 9** Sketch the graph of

$$\text{(a)}\ y = x^2 - 2x - 2 \qquad \text{(b)}\ y = -x^2 + 4x - 5$$

Solution (a). The equation is of form (7) with $a = 1$, $b = -2$, and $c = -2$, so by (8) the x-coordinate of the vertex is

$$x = -\frac{b}{2a} = 1$$

Using this value and two additional values on each side, we obtain Figure G.8.

Solution (b). The equation is of form (7) with $a = -1$, $b = 4$, and $c = -5$, so by (8) the x-coordinate of the vertex is

$$x = -\frac{b}{2a} = 2$$

Using this value and two additional values on each side, we obtain the table and graph in Figure G.9. ◀

Quite often the intercepts of a parabola $y = ax^2 + bx + c$ are important to know. The y-intercept, $y = c$, results immediately by setting $x = 0$. However, in order to obtain the x-intercepts, if any, we must set $y = 0$ and then solve the resulting quadratic equation $ax^2 + bx + c = 0$.

▶ **Example 10** Solve the inequality

$$x^2 - 2x - 2 > 0$$

Solution. Because the left side of the inequality does not have readily discernible factors, the test-point method illustrated in Example 4 of Appendix A is not convenient to use. Instead, we will give a graphical solution. The given inequality is satisfied for those values of x where the graph of $y = x^2 - 2x - 2$ is above the x-axis. From Figure G.8 those are the values of x to the left of the smaller intercept or to the right of the larger intercept. To find these intercepts we set $y = 0$ to obtain

$$x^2 - 2x - 2 = 0$$

Solving by the quadratic formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{12}}{2} = 1 \pm \sqrt{3}$$

Thus, the x-intercepts are

$$x = 1 + \sqrt{3} \approx 2.7 \quad \text{and} \quad x = 1 - \sqrt{3} \approx -0.7$$

and the solution set of the inequality is

$$(-\infty, 1 - \sqrt{3}) \cup (1 + \sqrt{3}, +\infty)$$ ◀

Note that the decimal approximations of the intercepts calculated in the preceding example agree with the graph in Figure G.8. Observe, however, that we used the exact values of the intercepts to express the solution. The choice of exact versus approximate values is often a matter of judgment that depends on the purpose for which the values are to be used. Numerical approximations often provide a sense of size that exact values do not, but they can introduce severe errors if not used with care.

Example 11 From Figure G.9 we see that the parabola $y = -x^2 + 4x - 5$ has no x-intercepts. This can also be seen algebraically by solving for the x-intercepts. Setting $y = 0$ and solving the resulting equation

$$-x^2 + 4x - 5 = 0$$

by the quadratic formula yields

$$y = \frac{-4 \pm \sqrt{16 - 20}}{-2} = 2 \pm i$$

Because the solutions are not real numbers, there are no x-intercepts. ◀

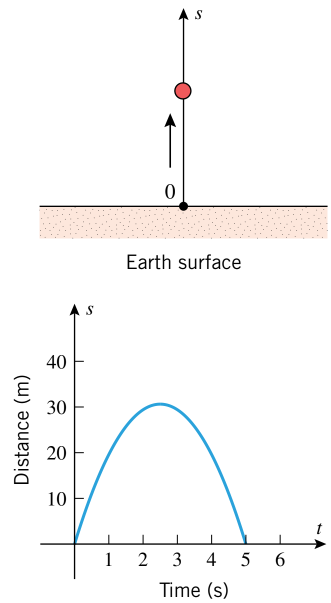


Figure G.10

Example 12 A ball is thrown straight up from the surface of the Earth at time $t = 0$ s with an initial velocity of 24.5 m/s. If air resistance is ignored, it can be shown that the distance s (in meters) of the ball above the ground after t seconds is given by

$$s = 24.5t - 4.9t^2 \tag{9}$$

(a) Graph s versus t, making the t-axis horizontal and the s-axis vertical.

(b) How high does the ball rise above the ground?

Solution (a). Equation (9) is of form (7) with $a = -4.9$, $b = 24.5$, and $c = 0$, so by (8) the t-coordinate of the vertex is

$$t = -\frac{b}{2a} = -\frac{24.5}{2(-4.9)} = 2.5 \text{ s}$$

and consequently the s-coordinate of the vertex is

$$s = 24.5(2.5) - 4.9(2.5)^2 = 30.625 \text{ m}$$

The factored form of (9) is

$$s = 4.9t(5 - t)$$

so the graph has t-intercepts $t = 0$ and $t = 5$. From the vertex and the intercepts we obtain the graph shown in Figure G.10.

Solution (b). From the s-coordinate of the vertex we deduce that the ball rises 30.625 m above the ground. ◀

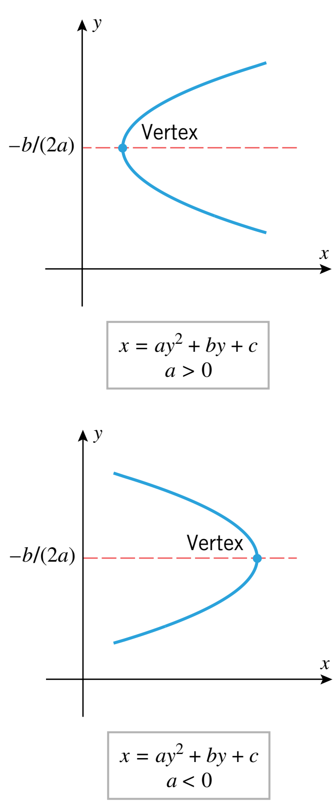


Figure G.11

THE GRAPH of $x = ay^2 + by + c$

If x and y are interchanged in (7), the resulting equation,

$$x = ay^2 + by + c$$

is called a ***quadratic equation in y***. The graph of such an equation is a parabola with its line of symmetry parallel to the x-axis and its vertex at the point with y-coordinate $y = -b/(2a)$ (Figure G.11). Some problems relating to such equations appear in the exercises.

EXERCISE SET G

1. Where in this section did we use the fact that the same scale was used on both coordinate axes?

2–5 Find
(a) the distance between A and B
(b) the midpoint of the line segment joining A and B.

2. $A(2, 5)$, $B(-1, 1)$ **3.** $A(7, 1)$, $B(1, 9)$
4. $A(2, 0)$, $B(-3, 6)$ **5.** $A(-2, -6)$, $B(-7, -4)$

6–10 Use the distance formula to solve the given problem.

6. Prove that $(1, 1)$, $(-2, -8)$, and $(4, 10)$ lie on a straight line.

7. Prove that the triangle with vertices $(5, -2)$, $(6, 5)$, $(2, 2)$ is isosceles.

8. Prove that $(1, 3)$, $(4, 2)$, and $(-2, -6)$ are vertices of a right triangle and then specify the vertex at which the right angle occurs.

9. Prove that $(0, -2)$, $(-4, 8)$, and $(3, 1)$ lie on a circle with center $(-2, 3)$.

10. Prove that for all values of t the point $(t, 2t - 6)$ is equidistant from $(0, 4)$ and $(8, 0)$.

11. Find k, given that $(2, k)$ is equidistant from $(3, 7)$ and $(9, 1)$.

12. Find x and y if $(4, -5)$ is the midpoint of the line segment joining $(-3, 2)$ and (x, y).

13–14 Find an equation of the given line.

13. The line is the perpendicular bisector of the line segment joining $(2, 8)$ and $(-4, 6)$.

14. The line is the perpendicular bisector of the line segment joining $(5, -1)$ and $(4, 8)$.

15. Find the point on the line $4x - 2y + 3 = 0$ that is equidistant from $(3, 3)$ and $(7, -3)$. [*Hint:* First find an equation of the line that is the perpendicular bisector of the line segment joining $(3, 3)$ and $(7, -3)$.]

16. Find the distance from the point $(3, -2)$ to the line
(a) $y = 4$ (b) $x = -1$.

17. Find the distance from $(2, 1)$ to the line $4x - 3y + 10 = 0$. [*Hint:* Find the foot of the perpendicular dropped from the point to the line.]

18. Find the distance from $(8, 4)$ to the line $5x + 12y - 36 = 0$. [*Hint:* See the hint in Exercise 17.]

19. Use the method described in Exercise 17 to prove that the distance d from (x_0, y_0) to the line $Ax + By + C = 0$ is

$$d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$

20. Use the formula in Exercise 19 to solve Exercise 17.

21. Use the formula in Exercise 19 to solve Exercise 18.

22. Prove: For any triangle, the perpendicular bisectors of the sides meet at a point. [*Hint:* Position the triangle with one vertex on the y-axis and the opposite side on the x-axis, so that the vertices are $(0, a)$, $(b, 0)$, and $(c, 0)$.]

23–24 Find the center and radius of each circle.

23. (a) $x^2 + y^2 = 25$
(b) $(x - 1)^2 + (y - 4)^2 = 16$
(c) $(x + 1)^2 + (y + 3)^2 = 5$
(d) $x^2 + (y + 2)^2 = 1$

24. (a) $x^2 + y^2 = 9$
(b) $(x - 3)^2 + (y - 5)^2 = 36$
(c) $(x + 4)^2 + (y + 1)^2 = 8$
(d) $(x + 1)^2 + y^2 = 1$

25–32 Find the standard equation of the circle satisfying the given conditions.

25. Center $(3, -2)$; radius $= 4$.

26. Center $(1, 0)$; diameter $= \sqrt{8}$.

27. Center $(-4, 8)$; circle is tangent to the x-axis.

28. Center $(5, 8)$; circle is tangent to the y-axis.

29. Center $(-3, -4)$; circle passes through the origin.

30. Center $(4, -5)$; circle passes through $(1, 3)$.

31. A diameter has endpoints $(2, 0)$ and $(0, 2)$.

32. A diameter has endpoints $(6, 1)$ and $(-2, 3)$.

33–44 Determine whether the equation represents a circle, a point, or no graph. If the equation represents a circle, find the center and radius.

33. $x^2 + y^2 - 2x - 4y - 11 = 0$

34. $x^2 + y^2 + 8x + 8 = 0$

35. $2x^2 + 2y^2 + 4x - 4y = 0$

36. $6x^2 + 6y^2 - 6x + 6y = 3$

37. $x^2 + y^2 + 2x + 2y + 2 = 0$

38. $x^2 + y^2 - 4x - 6y + 13 = 0$

39. $9x^2 + 9y^2 = 1$ **40.** $(x^2/4) + (y^2/4) = 1$

41. $x^2 + y^2 + 10y + 26 = 0$

42. $x^2 + y^2 - 10x - 2y + 29 = 0$

43. $16x^2 + 16y^2 + 40x + 16y - 7 = 0$

44. $4x^2 + 4y^2 - 16x - 24y = 9$

45. Find an equation of
(a) the bottom half of the circle $x^2 + y^2 = 16$
(b) the top half of the circle $x^2 + y^2 + 2x - 4y + 1 = 0$.

46. Find an equation of
(a) the right half of the circle $x^2 + y^2 = 9$
(b) the left half of the circle $x^2 + y^2 - 4x + 3 = 0$.

47. Graph
(a) $y = \sqrt{25 - x^2}$ (b) $y = \sqrt{5 + 4x - x^2}$.

48. Graph
(a) $x = -\sqrt{4 - y^2}$ (b) $x = 3 + \sqrt{4 - y^2}$.

49. Find an equation of the line that is tangent to the circle
$$x^2 + y^2 = 25$$
at the point (3, 4) on the circle.

50. Find an equation of the line that is tangent to the circle at the point P on the circle
(a) $x^2 + y^2 + 2x = 9;\ P(2, -1)$
(b) $x^2 + y^2 - 6x + 4y = 13;\ P(4, 3)$.

51. For the circle $x^2 + y^2 = 20$ and the point $P(-1, 2)$:
(a) Is P inside, outside, or on the circle?
(b) Find the largest and smallest distances between P and points on the circle.

52. Follow the directions of Exercise 51 for the circle
$$x^2 + y^2 - 2y - 4 = 0$$
and the point $P\left(3, \frac{5}{2}\right)$.

53. Referring to the accompanying figure, find the coordinates of the points T and T', where the lines L and L' are tangent to the circle of radius 1 with center at the origin.

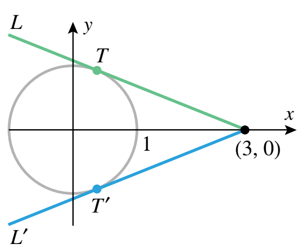


Figure Ex-53

54. A point (x, y) moves so that its distance to (2, 0) is $\sqrt{2}$ times its distance to (0, 1).
(a) Show that the point moves along a circle.
(b) Find the center and radius.

55. A point (x, y) moves so that the sum of the squares of its distances from (4, 1) and (2, −5) is 45.
(a) Show that the point moves along a circle.
(b) Find the center and radius.

56. Find all values of c for which the system of equations
$$\begin{cases} x^2 - y^2 = 0 \\ (x - c)^2 + y^2 = 1 \end{cases}$$
has 0, 1, 2, 3, or 4 solutions. [*Hint:* Sketch a graph.]

57–70 Graph the parabola and label the coordinates of the vertex and the intersections with the coordinate axes.

57. $y = x^2 + 2$
58. $y = x^2 - 3$
59. $y = x^2 + 2x - 3$
60. $y = x^2 - 3x - 4$
61. $y = -x^2 + 4x + 5$
62. $y = -x^2 + x$
63. $y = (x - 2)^2$
64. $y = (3 + x)^2$
65. $x^2 - 2x + y = 0$
66. $x^2 + 8x + 8y = 0$
67. $y = 3x^2 - 2x + 1$
68. $y = x^2 + x + 2$
69. $x = -y^2 + 2y + 2$
70. $x = y^2 - 4y + 5$

71. Find an equation of
(a) the right half of the parabola $y = 3 - x^2$
(b) the left half of the parabola $y = x^2 - 2x$.

72. Find an equation of
(a) the upper half of the parabola $x = y^2 - 5$
(b) the lower half of the parabola $x = y^2 - y - 2$.

73. Graph
(a) $y = \sqrt{x + 5}$ (b) $x = -\sqrt{4 - y}$.

74. Graph
(a) $y = 1 + \sqrt{4 - x}$ (b) $x = 3 + \sqrt{y}$.

75. If a ball is thrown straight up with an initial velocity of 32 ft/s, then after t seconds the distance s above its starting height, in feet, is given by $s = 32t - 16t^2$.
(a) Graph this equation in a ts-coordinate system (t-axis horizontal).
(b) At what time t will the ball be at its highest point, and how high will it rise?

76. A rectangular field is to be enclosed with 500 ft of fencing along three sides and by a straight stream on the fourth side. Let x be the length of each side perpendicular to the stream, and let y be the length of the side parallel to the stream.
(a) Express y in terms of x.
(b) Express the area A of the field in terms of x.
(c) What is the largest area that can be enclosed?

77. A rectangular plot of land is to be enclosed using two kinds of fencing. Two opposite sides will have heavy-duty fencing costing \$3/ft, and the other two sides will have standard fencing costing \$2/ft. A total of \$600 is available for the fencing. Let x be the length of each side with the heavy-duty fencing, and let y be the length of each side with the standard fencing.
(a) Express y in terms of x.
(b) Find a formula for the area A of the rectangular plot in terms of x.
(c) What is the largest area that can be enclosed?

78. (a) By completing the square, show that the quadratic equation $y = ax^2 + bx + c$ can be rewritten as
$$y = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$
if $a \neq 0$.
(b) Use the result in part (a) to show that the graph of the quadratic equation $y = ax^2 + bx + c$ has its high point at $x = -b/(2a)$ if $a < 0$ and its low point there if $a > 0$.

79–80 Solve the given inequality.

79. (a) $2x^2 + 5x - 1 < 0$ (b) $x^2 - 2x + 3 > 0$
80. (a) $x^2 + x - 1 > 0$ (b) $x^2 - 4x + 6 < 0$

81. At time $t = 0$ a ball is thrown straight up from a height of 5 ft above the ground. After t seconds its distance s, in feet, above the ground is given by $s = 5 + 40t - 16t^2$.
(a) Find the maximum height of the ball above the ground.

(b) Find, to the nearest tenth of a second, the time when the ball strikes the ground.

(c) Find, to the nearest tenth of a second, how long the ball will be more than 12 ft above the ground.

82. Find all values of x at which points on the parabola $y = x^2$ lie below the line $y = x + 3$.