CALCULUS 3

November 11, 2009

3rd TEST

YOUR NAME:

- ◯ **001** J. SANDERS .... (8AM)
- ◯ **002** J. KISH .... (9AM)
- ◯ **003** E. WITTENBORN .... (10AM)
- ◯ **004** A. SPINA .... (11AM)
- ◯ **005** A. SPINA .... (12PM)
- ◯ **006** M. STACKPOLE .... (1PM)
- ◯ **007** C. MESA .... (3PM)

SHOW ALL YOUR WORK

final answers without any supporting work
will receive no credit *even if they are right!*

No calculators allowed.
No cheat-sheets allowed.

Partial credit will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your test contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** one that is **counted** for grading!

DO NOT WRITE INSIDE THIS BOX!

problem	points	score
1	10 pts	
2	10 pts	
3	15 pts	
4	15 pts	
5	15 pts	
6	10 pts	
7	10 pts	
8	15 pts	
TOTAL	100 pts	

1. **[10 pts]** Find the absolute extrema of the function $f(x, y) = x^2 + 2y^2 - x$ on the closed and bounded disk $R : x^2 + y^2 \leq 4$.

SOLUTION:

- **Interior of R.**

$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x - 1 = 0 \\ \dfrac{\partial f}{\partial y} = 4y = 0 \end{cases} \quad \Rightarrow \quad (x_1, y_1)_c = \left(\frac{1}{2}, 0\right).$$

- **Boundary of R.**
There are two more or less obvious ways in which we can approach this part, either parametrize the boundary using x as a parameter, or parametrize it using θ as a parameter. The first approach will prove to be the simplest. $g(x) = f\big(x, y(x)\big)$,

$$\begin{cases} f(x, y) = x^2 + 2y^2 - x \\ x^2 + y^2 = 4 \end{cases}$$

$$\Rightarrow \quad g(x) = f\big(x, y(x)\big) = x^2 + 2\left(4 - x^2\right) - x = 8 - x - x^2$$

The critical points of $g(x)$ are found as

$$g'(x) = -1 - 2x = 0 \quad \Rightarrow \quad x_c = -\frac{1}{2} \quad \Rightarrow \quad y_c = \sqrt{4 - \left(-\frac{1}{2}\right)^2} = \pm\frac{\sqrt{15}}{2},$$

and the critical point in the xy–plane is

$$(x_2, y_2)_c = \left(\frac{1}{2}, +\frac{\sqrt{15}}{2}\right), \qquad (x_3, y_3)_c = \left(-\frac{1}{2}, -\frac{\sqrt{15}}{2}\right).$$

- **Classification.**

(x, y)	$f(x, y)$	classification
$\left(\frac{1}{2}, 0\right)$	$-\frac{1}{4}$	absolute minimum
$\left(-\frac{1}{2}, \pm\frac{\sqrt{15}}{2}\right)$	$+\frac{33}{4}$	absolute maximum

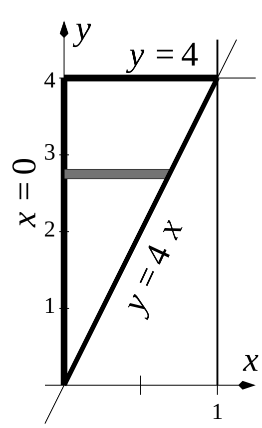
2. **[10 pts]** Evaluate the double integral

$$\iint_R e^{-y^2}\, \mathrm{d}A$$

where R is the region bounded by the lines $x = 0$, $y = 4$, and $y = 4x$.

SOLUTION:

We have to chose an order of integration. Integrating first over the variable y is out of the question, because this would lead to *error functions*. We integrate first over x,



$$\begin{aligned} \iint_R e^{-y^2}\, \mathrm{d}A &= \int_{y=0}^{4} \int_{x=0}^{y/4} e^{-y^2}\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{y=0}^{4} e^{-y^2} \frac{y}{4}\, \mathrm{d}y \\ &= \left[-\frac{1}{8} e^{-y^2}\right]_{y=0}^{4} \\ &= \frac{1}{8}\left(1 - e^{-16}\right). \end{aligned}$$

3. **[15 pts]** Find the area of the region in the first quadrant bounded by $r = 1$ and $r = \sin 2\theta$, with $\pi/4 \leq \theta \leq \pi/2$.
NOTE: The first curve is a circle and the second a polar rose. It is not really necessary to graph those curves to find the limits of this particular integral.

SOLUTION:

The boundaries of the region, in *polar coordinates* are

$$\begin{array}{ccccc} \frac{\pi}{4} & \leq & \theta & \leq & \frac{\pi}{2} \\ \sin 2\theta & \leq & r & \leq & 1 \end{array}$$

and the area is computed as follows,

$$\begin{aligned}
\iint_R \mathrm{d}A &= \int_{\theta=\pi/4}^{\pi/2} \int_{r=\sin 2\theta}^{1} r \ \mathrm{d}r \ \mathrm{d}\theta \\
&= \int_{\theta=\pi/4}^{\pi/2} \left[\frac{r^2}{2}\right]_{r=\sin 2\theta}^{1} \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=\pi/4}^{\pi/2} \left(1-\sin^2 2\theta\right) \ \mathrm{d}\theta \\
&= \frac{1}{2}\int_{\theta=\pi/4}^{\pi/2} \left(1-\frac{1-\cos 4\theta}{2}\right) \ \mathrm{d}\theta \\
&= \frac{1}{4}\int_{\theta=\pi/4}^{\pi/2} (1+\cos 4\theta) \ \mathrm{d}\theta \\
&= \frac{1}{4}\left[\theta+\frac{\sin 4\theta}{4}\right]_{\theta=\pi/4}^{\pi/2} \\
&= \frac{\pi}{16}\,.
\end{aligned}$$

4. **[15 pts]** Find the area of the surface of the portion of the paraboloid $z = 1 - x^2 - y^2$ that is above the xy–plane.

SOLUTION:

$$\begin{aligned}
S &= \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ \mathrm{d}A \\
&= \iint_R \sqrt{1+(-2x)^2+(-2y)^2}\ \mathrm{d}A \\
&= \iint_R \sqrt{1+4\left(x^2+y^2\right)}\ \mathrm{d}A \\
&= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \sqrt{1+4r^2}\ r\ \mathrm{d}r\ \mathrm{d}\theta \\
&= \left(\int_{r=0}^{1} \sqrt{1+4r^2}\ r\ \mathrm{d}r\right)\left(\int_{\theta=0}^{2\pi} \mathrm{d}\theta\right) \\
&= \left[\frac{1}{12}\left(1+4r^2\right)^{3/2}\right]_{r=0}^{1}\ \left[\theta\right]_{\theta=0}^{2\pi} \\
&= \frac{5\sqrt{5}-1}{12}\,2\pi \ = \ \frac{5\sqrt{5}-1}{6}\,\pi\,.
\end{aligned}$$

5. **[15 pts]** A surface S is given, in polar coordinates, by the equation $z = f(r, \theta)$, with (r, θ) in some region R in the $r\theta$–parameter space. Write the position vector $\mathbf{r}$ in terms of the polar variables r and θ, that is, $\mathbf{r} =< x(r,\theta), y(r,\theta), z(r,\theta) >$, compute

$$\left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\|$$

and use it to complete the expression below for surface area

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \boxed{}}\; r \, \mathrm{d}r \, \mathrm{d}\theta$$

Show your work!

SOLUTION:

$$\mathbf{r}(r,\theta) = \langle r\cos\theta\,,\, r\sin\theta\,,\, f(r,\theta)\rangle$$

$$\frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & \dfrac{\partial f}{\partial r} \\ -r\sin\theta & r\cos\theta & \dfrac{\partial f}{\partial \theta} \end{vmatrix} = \left\langle \sin\theta \frac{\partial f}{\partial \theta} - r\cos\theta \frac{\partial f}{\partial r}\,,\, -r\sin\theta \frac{\partial f}{\partial r} - \cos\theta \frac{\partial f}{\partial \theta}\,,\, r \right\rangle$$

$$\left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\|^2 = \left(\frac{\partial f}{\partial \theta}\right)^2 + r^2 \left(\frac{\partial f}{\partial r}\right)^2 + r^2$$

$$\Rightarrow \quad \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| = \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2}\; r$$

Therefore,

$$S = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial r} \times \frac{\partial \mathbf{r}}{\partial \theta} \right\| \mathrm{d}A = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \boxed{\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2}}\; r \, \mathrm{d}r \, \mathrm{d}\theta$$

6. **[10 pts]** Complete the limits of the following iterated integral,

$$\int_{y=0}^{4} \int_{z=y}^{8-y} \int_{x=0}^{\sqrt{4-y}} f(x,y,z)\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}y = \int_{x=\boxed{}}^{\boxed{}} \int_{y=\boxed{}}^{\boxed{}} \int_{z=\boxed{}}^{\boxed{}} f(x,y,z)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x\,.$$

Show your work!

SOLUTION:

The boundaries of x do not depend on z, and those of z are already in the form

$$z_{\min}(y) \le z \le z_{\max}(y)$$

so these do not need to be touched. (Draw the graph.) For the limits in the xy–plane, just sketch the region and you get the first-quadrant portion of the inner part of the parabola $y = 4 - x^2$,

$$\begin{cases} 0 \le x \le \sqrt{4-y} \\ y \le z \le 8-y \\ 0 \le y \le 4 \end{cases} \quad \Rightarrow \quad \begin{cases} y \le z \le 8-y \\ 0 \le y \le 4-x^2 \\ 0 \le x \le 2 \end{cases}$$

Therefore

$$\int_{y=0}^{4}\int_{z=y}^{8-y}\int_{x=0}^{\sqrt{4-y}} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y = \int_{x=\boxed{0}}^{\boxed{2}}\int_{y=\boxed{0}}^{\boxed{4-x^2}}\int_{z=\boxed{y}}^{\boxed{8-y}} f(x,y,z)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\,.$$

7. **[10 pts]** Spherical coordinates are used to compute the volume of the solid between the sphere $x^2 + y^2 + z^2 = 9$ and outside the cone $z = \sqrt{x^2+y^2}$, and above the xy–plane. Complete, but **do not evaluate** the integral below,

$$V = \int_{\theta=\boxed{}}^{\boxed{}}\int_{\phi=\boxed{}}^{\boxed{}}\int_{\rho=\boxed{}}^{\boxed{}} \boxed{}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Show your work!

SOLUTION:

In spherical coordinates the bounding surfaces of the solid are given by

$$\begin{aligned} x^2+y^2+z^2 = 9 \quad &\longrightarrow \quad \rho = 3 \\ z = \sqrt{x^2+y^2} \quad &\longrightarrow \quad \phi = \pi/4 \\ z = 0 \quad &\longrightarrow \quad \phi = \pi/2 \end{aligned}$$

Therefore, the required integral is

$$V = \int_{\theta=\boxed{0}}^{\boxed{2\pi}}\int_{\phi=\boxed{\pi/4}}^{\boxed{\pi/2}}\int_{\rho=\boxed{0}}^{\boxed{3}} \boxed{\rho^2\sin\phi}\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

8. **[15 pts]** Use the transformation $u = y/x$, $v = xy$ to find

$$\iint_R xy^3 \,\mathrm{d}A$$

over the region R in the first quadrant enclosed by $y = x$, $y = 3x$, $xy = 1$, $xy = 4$.

SOLUTION:

$$\iint_{R_{xy}} f(x,y)\,\mathrm{d}A_{xy} = \iint_{R_{uv}} f\big(x(u,v),y(u,v)\big)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}A_{uv}$$

$$R_{xy} : \begin{cases} y = x\,, & y = 3x \\ xy = 1\,, & xy = 4 \end{cases} \quad\Rightarrow\quad R_{xy} : \begin{cases} y/x = 1\,, & y/x = 3 \\ xy = 1\,, & xy = 4 \end{cases}$$

$$\Rightarrow \quad R_{xy} : \begin{cases} 1 \le y/x \le 3 \\ 1 \le xy \le 4 \end{cases} \quad\Rightarrow\quad R_{uv} : \begin{cases} 1 \le u \le 3 \\ 1 \le v \le 4 \end{cases}$$

$$\begin{cases} u = y/x \\ v = xy \end{cases} \quad\Rightarrow\quad \begin{cases} x = u^{-1/2}v^{1/2} \\ y = u^{1/2}v^{1/2} \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2}\,u^{-3/2}v^{1/2} & \frac{1}{2}\,u^{-1/2}v^{-1/2} \\ \frac{1}{2}\,u^{-1/2}v^{1/2} & \frac{1}{2}\,u^{1/2}v^{-1/2} \end{vmatrix} = -\frac{1}{2u}\,.$$

$$f\big(x(u,v),y(u,v)\big) = x(u,v)\big(y(y,v)\big)^3 = \frac{v^{1/2}}{u^{1/2}}\,\Big(u^{1/2}v^{1/2}\Big)^3 = uv^2\,.$$

$$\begin{aligned}
\iint\limits_{R_{xy}} f(x,y)\;\mathrm{d}A_{xy} &= \iint\limits_{R_{uv}} f\big(x(u,v),y(u,v)\big)\,\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\;\mathrm{d}A_{uv} \\
&= \iint\limits_{R_{uv}} uv^2\left|-\frac{1}{2u}\right|\;\mathrm{d}A_{uv} \\
&= \frac{1}{2}\int_{u=1}^{3}\int_{v=1}^{4} v^2\;\mathrm{d}v\;\mathrm{d}u \\
&= \frac{1}{2}\left(\int_{u=1}^{3}\mathrm{d}u\right)\left(\int_{v=1}^{4} v^2\;\mathrm{d}v\right) \\
&= \frac{1}{2}\,[u]_{u=1}^{3}\,\big[v^3/3\big]_{v=1}^{4} \\
&= \frac{1}{2}\,(3-1)\left(\frac{4^3-1}{3}\right) \\
&= 21\,.
\end{aligned}$$