

# CALCULUS 3

October 14, 2009

## 2nd TEST

**YOUR NAME:**

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|---|--|
| <input type="radio"/> <b>001</b> J. SANDERS ..... (8AM)<br><input type="radio"/> <b>002</b> J. KISH ..... (9AM)<br><input type="radio"/> <b>003</b> E. WITTENBORN ..... (10AM)<br><input type="radio"/> <b>004</b> A. SPINA .....(11AM) | <input type="radio"/> <b>005</b> A. SPINA .....(12PM)<br><input type="radio"/> <b>006</b> M. STACKPOLE ..... (1PM)<br><input type="radio"/> <b>007</b> C. MESA ..... (3PM) |
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### SHOW ALL YOUR WORK

final answers without any supporting work  
will receive no credit *even if they are right!*

No calculators allowed.  
No cheat-sheets allowed.

**Partial credit** will be given for any **reasonable amount of work pointing in the right direction** towards the solution of your problem. You will not get any partial credit for memorizing formulas and not knowing how to use them, or for anything you write that is not directly related to the solution of your problem.

If your tests contains **more than one solution or answer** to a problem or part of a problem, and one of them is wrong, then it will be **the wrong one** the one that **counts** for your grading!

**DO NOT WRITE INSIDE THIS BOX!**

problem	points	score
<b>1</b>	12 pts	
<b>2</b>	12 pts	
<b>3</b>	13 pts	
<b>4</b>	12 pts	
<b>5</b>	13 pts	
<b>6</b>	13 pts	
<b>7</b>	13 pts	
<b>8</b>	12 pts	
<b>TOTAL</b>	100 pts	

1. [12 pts] Find the length of the curve

$$\mathbf{r}(t) = \langle 2t^{3/2}, \sin(2t), \cos(2t) \rangle$$

for  $0 \leq t \leq 1$ .

**SOLUTION:**

The length of a parametric curve  $\mathbf{r}(t)$ , between  $t_i$  and  $t_f$  is given by

$$L = \int_{t=t_i}^{t_f} \left\| \frac{d\mathbf{v}(t)}{dt} \right\| dt$$

In our case we have

$$\frac{d\mathbf{v}(t)}{dt} = \langle 3t^{1/2}, 2\cos(2t), -2\sin(2t) \rangle$$

and

$$\begin{aligned} L &= \int_{t=0}^1 \sqrt{(3t^{1/2})^2 + (2\cos(2t))^2 + (-2\sin(2t))^2} dt \\ &= \int_{t=0}^1 \sqrt{9t + 4\cos^2(2t) + 4\sin^2(2t)} dt \\ &= \int_{t=0}^1 \sqrt{9t + 4} dt \\ &= 3 \int_{t=0}^1 \sqrt{t + 4/9} dt \\ &= 3 \left[ \frac{2}{3} \left( t + \frac{4}{9} \right)^{3/2} \right]_{t=0}^1 \\ &= 3 \frac{2}{3} \left[ \left( 1 + \frac{4}{9} \right)^{3/2} - \left( \frac{4}{9} \right)^{3/2} \right] \\ &= 2 \left[ \left( \frac{13}{9} \right)^{3/2} - \left( \frac{4}{9} \right)^{3/2} \right] \\ &= \frac{2}{27} (13^{3/2} - 8). \end{aligned}$$

2. [12 pts] Find and sketch the domain of the function  $f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$ .

**SOLUTION:**

$$\begin{array}{llll} \text{the square root requires} & x^2 + y^2 - 1 \geq 0 & \Rightarrow & x^2 + y^2 \geq 1, \\ \text{the logarithm requires} & 4 - x^2 - y^2 > 0 & \Rightarrow & x^2 + y^2 < 4. \end{array}$$

Therefore,

$$1 \leq x^2 + y^2 < 4.$$

3. [13 pts] Determine if

$$f(x, y) = \begin{cases} (x^2 + y^2) \ln(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous at  $(0, 0)$ .

**SOLUTION:**

Computing the limit of  $f$  when  $(x, y) \rightarrow (0, 0)$  we obtain

$$\begin{aligned} \lim_{(x,y) \rightarrow (0,0)} f(x, y) &= \lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2) \ln(x^2 + y^2) \\ &= \lim_{r \rightarrow 0} r^2 \ln(r^2) \\ &= \lim_{r \rightarrow 0} \frac{\ln(r^2)}{1/r^2} \\ &= \lim_{r \rightarrow 0} \frac{2r/r^2}{-2/r^3} \\ &= \lim_{r \rightarrow 0} (-r^2) \\ &= 0, \end{aligned}$$

where we have used L'Hôpital's rule in evaluating the  $r$ -limit. The limit of the function for  $(x, y) \rightarrow (0, 0)$  equal the value of the the function at  $(0, 0)$ , therefore  $f$  is **continuous** at  $(0, 0)$ .

4. [12 pts] Find the first partial derivatives and all second partial derivatives of  $f(x, y) = xe^{xy^2}$ .  
**Simplify your answers!**

**SOLUTION:**

The first derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x} &= (1 + xy^2) e^{xy^2}, \\ \frac{\partial f}{\partial y} &= 2x^2 y e^{xy^2}. \end{aligned}$$

The second derivatives are

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = y^2 (2 + xy^2) e^{xy^2}, \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = 2xy (2 + xy^2) e^{xy^2}, \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = 2x^2 (1 + 2xy^2) e^{xy^2}. \end{aligned}$$

5. [13 pts] An object's specific gravity  $S$  can be found using the formula

$$S = \frac{A}{A - W},$$

where  $A$  is the number of pounds the object weighs in air and  $W$  is the number of pounds the object weighs in water. The weight in air  $A$  is measured to be 10 pounds, with a maximum possible error in the measurement of  $\Delta A = 0.1$  pounds. The weight in water  $W$  is measured to be 8 pounds, with a maximum possible error in the measurement of  $\Delta W = 0.08$  pounds. Use differentials to estimate the maximum possible error in calculating specific gravity.

**SOLUTION:**

$$\begin{aligned} S &= S(A, W) = \frac{A}{A - W} \\ \Delta S &\approx \frac{\partial S}{\partial A} \Delta A + \frac{\partial S}{\partial W} \Delta W \\ &= \frac{\partial}{\partial A} \left[ \frac{A}{A - W} \right] \Delta A + \frac{\partial}{\partial W} \left[ \frac{A}{A - W} \right] \Delta W \\ &= -\frac{W}{(A - W)^2} \Delta A + \frac{A}{(A - W)^2} \Delta W \\ &= \frac{A \Delta W - W \Delta A}{(A - W)^2} \end{aligned}$$

Since

$$|\Delta S| \approx \left| \frac{A\Delta W - W\Delta A}{(A - W)^2} \right| \leq \frac{A|\Delta W| + W|\Delta A|}{(A - W)^2},$$

the maximum possible error is given then by

$$\varepsilon_{\max} = \frac{A|\Delta W| + W|\Delta A|}{(A - W)^2}.$$

For the data given, the maximum possible error will then be

$$\varepsilon_{\max} = \frac{10 \times 0.08 + 8 \times 0.1}{(10 - 8)^2} = \frac{0.8 + 0.8}{4} = 0.4.$$

6. [13 pts] Use chain rule and implicit differentiation to express  $\partial w/\partial r$  and  $\partial w/\partial s$  in terms of  $r$  and  $s$  if

$$w^2 = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

**SOLUTION:**

$$\begin{aligned} 2w \frac{\partial w}{\partial r} &= 2x \frac{\partial x}{\partial r} + 2y \frac{\partial y}{\partial r} \\ &= 2x + 2y. \end{aligned}$$

$$\begin{aligned} 2w \frac{\partial w}{\partial s} &= 2x \frac{\partial x}{\partial s} + 2y \frac{\partial y}{\partial s} \\ &= -2x + 2y. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{x + y}{w} \\ &= \frac{(r - s) + (r + s)}{\sqrt{(r - s)^2 + (r + s)^2}} \\ &= \frac{\sqrt{2}r}{\sqrt{r^2 + s^2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{-x + y}{w} \\ &= \frac{-(r - s) + (r + s)}{\sqrt{(r - s)^2 + (r + s)^2}} \\ &= \frac{\sqrt{2}s}{\sqrt{r^2 + s^2}}. \end{aligned}$$

7. [13 pts] Let  $f$  be a function of one variable, and let  $w = f(\rho)$ , where  $\rho = \sqrt{x^2 + y^2 + z^2}$ . Show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{df}{d\rho}\right)^2.$$

**SOLUTION:**

$$w = f(\rho) = f(\rho(x, y, z))$$

$$\frac{\partial f}{\partial x} = \left(\frac{df}{d\rho}\right) \left(\frac{\partial \rho}{\partial x}\right) = \left(\frac{df}{d\rho}\right) \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\frac{\partial f}{\partial y} = \left(\frac{df}{d\rho}\right) \left(\frac{\partial \rho}{\partial y}\right) = \left(\frac{df}{d\rho}\right) \left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\frac{\partial f}{\partial z} = \left(\frac{df}{d\rho}\right) \left(\frac{\partial \rho}{\partial z}\right) = \left(\frac{df}{d\rho}\right) \left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Squaring and adding,

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{df}{d\rho}\right)^2 \left(\frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2}\right).$$

Therefore

$$\boxed{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = \left(\frac{df}{d\rho}\right)^2}$$

8. [12 pts] A bug is located on the surface  $z = xe^{-y} + 3y$  at the point  $(1, 0, 1)$ . Assuming the positive  $z$ -direction is up, find a unit vector in the direction the bug should move for the steepest climb. What is the slope of the surface in the direction of steepest climb?

**SOLUTION:**

The surface is given by the graph of the function

$$f(x, y) = xe^{-y} + 3y.$$

The direction  $\mathbf{u}$  of maximum increase of the function (steepest climb) at the point  $P$  is given by the direction of its gradient evaluated at  $P$ , that is,  $\nabla f|_P$ , and the slope of the climb by the magnitude of this gradient.

We first compute the gradient of  $f$  at  $P$

$$\nabla f|_P = \langle e^{-y}, 3 - xe^{-y} \rangle|_P = \langle 1, 2 \rangle,$$

then the slope,

$$\|\nabla f|_P\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

and finally the direction,

$$\mathbf{u} = \frac{\nabla f|_P}{\|\nabla f|_P\|} = \frac{\langle 1, 2 \rangle}{\sqrt{5}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle.$$

The slope in the direction of steepest climb is given by the norm of the gradient vector, which we already computed, that is,  $\|\nabla f|_P\| = \sqrt{5}$ .