*Advances in Number Theory and Random Matrix Theory*

# *Random Matrix Theory and Heights of Polynomials*


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# *Ginibre's Ensembles*

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Ginibre's goal was to compute the joint eigenvalue probability density functions (JPDF) and the correlation functions for these ensembles.

# GinOE

GinOE is defined to be $\mathbb{R}^{N\times N}$ together with the probability measure $\nu$ given by

$$\nu(S) := \frac{1}{\nu(\mathbb{R}^{N\times N})} \int_S \exp\left\{-\frac{1}{2}\,\mathrm{Tr}(X^{\mathrm{T}}X)\right\} d\lambda(X),$$

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The eigenvalues of $X \in \mathbb{R}^{N\times N}$ are partitioned into real and complex conjugate pairs.

# *Eigenvalues in GinOE*

The space of eigenvalues of $\mathbb{R}^{N\times N}$ can be written as the disjoint union

$$\bigcup_{(L,M)} \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M,$$

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$$P_{N,0}(\boldsymbol{\alpha}) = \mathcal{C}_N^{-1} \frac{1}{N!} \left\{ \prod_{\ell=1}^{N} e^{-\alpha_\ell^2/2} \right\} \left\{ \prod_{j<k} |\alpha_k - \alpha_j| \right\}.$$

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Given $\boldsymbol{\alpha} \in \mathbb{C}^L$ and $\boldsymbol{\beta} \in \mathbb{C}^M$ we define

$$\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \det V^{\boldsymbol{\gamma}}$$

where

$$\boldsymbol{\gamma} := (\alpha_1, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M).$$

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**Conjecture 1 (S- 2005).** *There exist functions* $w_1 : \mathbb{R} \to \mathbb{R}$ *and* $w_2 : \mathbb{C} \to \mathbb{R}$ *such that*

$$P_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \propto \frac{|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})|}{L!M!} \times \left\{ \prod_{\ell=1}^{L} w_1(\alpha_\ell) \right\} \left\{ \prod_{m=1}^{M} w_2(\beta_m) \right\},$$

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**Theorem 1 (Lehmann & Sommers 1991, Edelman 1997).** *There exist functions* $w_1 : \mathbb{R} \to \mathbb{R}$ *and* $w_2 : \mathbb{C} \to \mathbb{R}$ *such that*

$$P_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{C}_N^{-1} \frac{|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})|}{L!M!} \times \left\{ \prod_{\ell=1}^{L} w_1(\alpha_\ell) \right\} \left\{ \prod_{m=1}^{M} w_2(\beta_m) \right\},$$

*where* $\mathcal{C}_N = 2^{N(N+1)/4} \prod_{n=1}^{N} \Gamma(n/2)$.

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**Theorem 1 (Lehmann & Sommers 1991, Edelman 1997).**

$$P_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \mathcal{C}_N^{-1} \frac{|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})|}{L!M!} \times \left\{ \prod_{\ell=1}^{L} e^{-\alpha_\ell^2/2} \right\} \left\{ \prod_{m=1}^{M} \operatorname{erfc}\left( \sqrt{2}\, |\operatorname{Im} \beta_m| \right) e^{-(\beta_m^2 + \overline{\beta_m}^2)/2} \right\},$$

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## *The Partial JPDFs*

We may simplify this by setting $w : \mathbb{C} \to \mathbb{R}$, where

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The two main difficulties in working with GinOE are now apparent:

1. The decomposition of the space of eigenvalues.
2. GinOE is a $\beta = 1$ ensemble.

# *Multiplicative Class Functions*

A function $\Psi : \mathbb{R}^{N\times N} \to \mathbb{R}$ will be called a *multiplicative class function* if

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- There exists a function $\psi : \mathbb{C} \to \mathbb{R}$ such that if $D$ is a diagonal matrix with entries $\gamma_1, \gamma_2, \ldots, \gamma_N$ then

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$$\langle \Psi \rangle := \frac{1}{(\sqrt{2\pi})^{N^2}} \int_{\mathbb{R}^{N\times N}} \Psi(X) \exp\left\{-\frac{1}{2}\operatorname{Tr}(X^{\mathrm{T}}X)\right\} d\lambda(X).$$

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$$\langle\Psi\rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L\times\mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \right\} \times |\Delta(\boldsymbol{\alpha},\boldsymbol{\beta})|\, d\lambda_L(\boldsymbol{\alpha})\, d\lambda_{2M}(\boldsymbol{\beta}),$$

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- $\lambda_{2M}$ is Lebesgue measure on $\mathbb{C}^M$.

# *Skew-symmetric inner products*

We introduce two skew-symmetric bilinear forms associated to $\Psi$ :

$$\langle P, Q \rangle_{\mathbb{R}} := \int_{\mathbb{R}^2} \varphi(\alpha_1)\varphi(\alpha_2) \; P(\alpha_1)Q(\alpha_2) \; \operatorname{sgn}(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2,$$

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and,

$$\langle P, Q \rangle_{\mathbb{C}} := -2i \int_{\mathbb{C}} \varphi(\beta)\varphi(\overline{\beta})\ P(\overline{\beta})Q(\beta)\ \operatorname{sgn}(\operatorname{Im}(\beta))\, d\lambda_2(\beta).$$

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By construction,

$$\langle Q, P \rangle_{\mathbb{R}} = -\langle P, Q \rangle_{\mathbb{R}} \qquad \text{and} \qquad \langle Q, P \rangle_{\mathbb{C}} = -\langle P, Q \rangle_{\mathbb{C}}.$$

**Theorem 2 (S- 2006).** *Let $N = 2J$, and let*

$$\mathbf{P} = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\}$$

*be a set of monic polynomials with $\deg P_n = n - 1$. Then,*

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \operatorname{Pf} U_{\mathbf{P}},$$

*the Pfaffian of $U_{\mathbf{P}}$, where $U_{\mathbf{P}}$ is the $N \times N$ matrix where,*

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When $N$ is odd the matrix $U_{\mathbf{P}}$ must be modified.

**Corollary 2.** *Let* $\langle P, Q\rangle = \langle P, Q\rangle_{\mathbb{R}} + \langle P, Q\rangle_{\mathbb{C}}$*, and let* $\mathbf{Q} = \{Q_1, Q_2, \ldots, Q_N\}$ *be a set of monic polynomials specified by*

$$\langle Q_{2k-1}, Q_{2j}\rangle = -\langle Q_{2j}, Q_{2k-1}\rangle = \delta_{kj}\mathfrak{M}_j$$

*and*

$$\langle Q_{2j}, Q_{2k}\rangle = \langle Q_{2j-1}, Q_{2k-1}\rangle = 0,$$

*Then,*

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \prod_{j=1}^{J} \mathfrak{M}_j.$$

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By Jensen's formula we also have

$$\mu(f) = \exp\left\{ \int_0^1 \log |f(e^{2\pi i\theta})| \, d\theta \right\}.$$

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- continuous on finite dimensional subspaces of $\mathbb{C}[x]$
- multiplicative: $\mu(fg) = \mu(f)\mu(g)$,
- absolutely homogeneous: $\mu(kf) = |k|\mu(f)$,
- positive definite: $\mu(f) = 0$ iff $f = 0$.

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> The following problem arises immediately. If $\epsilon$ is a positive quantity to find a polynomial of the form $f(x) = x^r + a_1 x^{r-1} + \ldots + a_r$ where the $a$'s are integers, such that the absolute value of those roots of $f$ which lie outside the unit circle lies between 1 and $1 + \epsilon$. (D.H. Lehmer 1933)

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In the same paper D.H. Lehmer states,

> We have not made an examination of all 10th degree symmetric polynomials, but a rather intensive search has failed to reveal a better polynomial than

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \quad \mu = 1.176\ldots$$

# *Smyth's Theorem*

A polynomial $f$ is said to be *reciprocal* if

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**Theorem 3 (Smyth 1971).** *If $g$ is an irreducible non-reciprocal polynomial in $\mathbb{Z}[x]$, and $g(x) \nmid x(x-1)$, then*

$$\mu(g) \geq \mu(x^3 - x - 1) = 1.324\ldots$$

Let $J$ be the integer part of $(N-1)/2$.

**Theorem 4 (Chern & Vaaler 2001).** *As* $T \to \infty$,

$$\#\{f \in \mathbb{Z}[x] : \deg f = N, \mu(f) \le T\} =$$

$$\left\{ \frac{2^{N-J+1}(N+1)^J}{J!} \prod_{j=1}^{J} \left( \frac{2j}{2j+1} \right)^{N+1-2j} \right\} T^{N+1} + O(T^N).$$

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# *The Connection with GinOE*

Recall our main result,

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \operatorname{Pf} U_{\mathbf{P}},$$

where $U_{\mathbf{P}}$ is the $N \times N$ matrix,

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The inspiration for this result came from the proof that as $T \to \infty$,

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There exists a *root function* $\phi : \mathbb{C} \to (0, \infty)$ such that

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Moreover, as $|\gamma| \to \infty$, $\phi(\gamma) \sim |\gamma|$.

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- The unit balls are compact but not convex — they are *star bodies*.
- If $\phi(\gamma) = \max\{1, |\gamma|\}$ then $\Phi$ is Mahler measure.
- The algebra of reciprocal polynomials is isomorphic to the algebra $\mathbb{C}[x + 1/x]$, if

$$\phi(\gamma) = \mu(x + 1/x - \gamma) = \max\left\{\left|\frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2}\right|\right\},$$

the $\Phi$ is the *reciprocal Mahler measure*.

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By the absolute homogeneity of $\Phi$,

$$\{\mathbf{a} \in \mathbb{R}^{N+1} : \Phi(\mathbf{a}) \leq T\} = T\mathcal{U}_N.$$

# *Asymptotic Estimates*

When $T$ is large, the volume of $T\mathcal{U}_N$ is a good estimate for the number of lattice points that it contains.

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# *Distribution Functions and Mellin Transforms*

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The Mellin transform of $f_N(\xi)$ is then

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change of variables:

$$y = \frac{1}{\xi} \qquad dy = -\frac{d\xi}{\xi^2}.$$

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integration by parts: (Lebesgue-Stieltjes)

$$u(\xi) = f_N(\xi) \qquad v = \frac{-1}{s}\xi^{-s}$$

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$$\nu(a, b] = f_N(b) - f_N(a) \qquad 0 < a < b < \infty.$$

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Thus we may replace the integral over $(0,\infty)$ with an integral over $\mathbb{R}^N$.

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$F_N(s)$ converges to an analytic function of $s$ in the half plane

$$\Re(s) > N.$$

# *Examples of Moment Functions*

As $T \to \infty$,

$$\#\{f \in \mathbb{Z}[x] : \deg f = N, \Phi(f) \leq T\} = \mathrm{vol}(\mathcal{U}_N)T^{N+1} + O(T^N).$$

# *Examples of Moment Functions*

As $T \to \infty$,

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**Theorem 8 (Chern & Vaaler 2001).** *Let* $J = \lfloor (N-1)/2 \rfloor$*, then*

$$F_N(\mu; s) = 2^N \left\{ \prod_{k=1}^{J} \left( \frac{2k}{2k+1} \right)^{N-2k} \right\} \left\{ \prod_{j=0}^{J} \frac{s}{s-(N-2j)} \right\}.$$

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**Theorem 8 (S- 2005).** *Let $\rho$ be the reciprocal Mahler measure. Then,*

$$F_N(\rho; s) = \frac{2^N}{N!} \left\{ \prod_{n=1}^{N} \left( \frac{2n}{2n-1} \right)^{N+1-n} \right\} \left\{ \prod_{j=0}^{J} \frac{s^2}{s^2-(N-2j)^2} \right\}.$$

# *A Change of Variables*

Next we will use the multiplicativity of $\Phi$ to rewrite $F_N(s)$ in terms of the roots of polynomials.

# *A Change of Variables*

The space of roots of degree $N$ polynomials in $\mathbb{R}[x]$ can be written as the disjoint union

$$\bigcup_{(L,M)} \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M,$$

where the union is over all pairs $(L, M)$ such that $L + 2M = N$.

# *A Change of Variables*

Given $(L, M)$, define

$$E_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \to \mathbb{R}^N$$

by $E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \mathbf{b}$ where

$$x^N + \sum_{n=1}^{N} b_n x^{N-n} = \prod_{\ell=1}^{L} (x - \alpha_\ell) \prod_{m=1}^{M} (x - \beta_m)(x - \overline{\beta_m}).$$

# A Change of Variables

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The degree of the map $E_{L,M}$ is $2^M M!L!$, and the images of the various $E_{L,M}$ are disjoint (except for a set of $\lambda_N$-measure 0).

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The degree of the map $E_{L,M}$ is $2^M M!L!$, and the images of the various $E_{L,M}$ are disjoint (except for a set of $\lambda_N$-measure 0). Moreover,

$$\widetilde{\Phi}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})) = \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta_m}).$$

# A Change of Variables

$$F_N(s) = \int_{\mathbb{R}^N} \widetilde{\Phi}(\boldsymbol{b})^{-s}\, d\lambda_N(\boldsymbol{b})$$

# A Change of Variables

$$F_N(s) = \sum_{(L,M)} \frac{1}{2^M M!L!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \widetilde{\Phi}(E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}))^{-s} \times \operatorname{Jac} E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta})\, d\lambda_L(\boldsymbol{\alpha})\, d\lambda_{2M}(\boldsymbol{\beta}).$$

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**Lemma 8.**

$$\operatorname{Jac} E_{L,M}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 2^M |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})|.$$

# A Change of Variables

$$F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta_m}) \right\}^{-s} \times |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}).$$

# The Connection

$$F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta_m}) \right\}^{-s} \times |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}).$$

# *The Connection*

$$F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta_m}) \right\}^{-s}$$
$$\times |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}).$$

And recall that

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \right\}$$
$$\times \, |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}),$$

where $\varphi(\gamma) = \psi(\gamma) \, e^{-\gamma^2/2} \left\{ \mathrm{erfc}(\sqrt{2}|\mathrm{Im}(\gamma)|) \right\}^{1/2}$.

## *The Idea of the Proof*

We wish to prove

$$\langle \Psi \rangle = \mathcal{C}_N^{-1}\,\mathrm{Pf}\,U_{\mathbf{P}},$$

where

$$U_{\mathbf{P}}[j,k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}.$$

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First we need to expand $|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})|$

# *Vandermonde Determinants*

$\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is given by

$$\det \begin{bmatrix} 1 & & 1 & 1 & 1 & & 1 & 1 \\ \alpha_1 & \cdots & \alpha_L & \overline{\beta_1} & \beta_1 & \cdots & \overline{\beta_M} & \beta_M \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots & \\ \alpha_1^{N-1} & \cdots & \alpha_L^{N-1} & \overline{\beta_1}^{N-1} & \beta_1^{N-1} & \cdots & \overline{\beta_M}^{N-1} & \beta_M^{N-1} \end{bmatrix}$$

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$$= \left\{ \prod_{j<k} (\alpha_k - \alpha_j) \right\} \prod_{\ell=1}^{L} \prod_{m=1}^{M} |\beta_m - \alpha_\ell|^2$$

$$\times \left\{ \prod_{m<n} |\beta_n - \beta_m|^2 \left|\beta_n - \overline{\beta_m}\right|^2 \right\} \prod_{m=1}^{M} 2i\Im(\beta_m).$$

Thus,

$$\left|\det V^{\boldsymbol{\alpha},\boldsymbol{\beta}}\right| = (-i)^M \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \operatorname{sgn} \Im(\beta_m) \right\} \det V^{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

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And, $\det V^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is given by

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Now, let $\mathbf{P} = \{P_1, P_2, \ldots, P_N\}$ be a set of monic polynomials $\deg P_n = n - 1$.

# *Vandermonde Determinants*

Thus,

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And, $\det V^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is given by

$$\det \begin{bmatrix} P_1(\alpha_1) & & P_1(\alpha_L) & P_1(\overline{\beta_1}) & P_1(\beta_1) & & P_1(\beta_M) \\ P_2(\alpha_1) & \cdots & P_2(\alpha_L) & P_2(\overline{\beta_1}) & P_2(\beta_1) & \cdots & P_2(\beta_M) \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ P_N(\alpha_1) & \cdots & P_N(\alpha_L) & P_N(\overline{\beta_1}) & P_N(\beta_1) & \cdots & P_N(\beta_M) \end{bmatrix}$$

Thus,

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Call this matrix $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

# *Vandermonde Determinants*

Thus,

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# *The Laplace Expansion of the Determinant*

We wish to enumerate the minors of $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

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$$\mathfrak{I}_L^N := \left\{\{1,2,\ldots,L\} \xrightarrow{\mathfrak{t}} \{1,2,\ldots,N\} \quad : \quad \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(L)\right\}.$$

To each $\mathfrak{t} \in \mathfrak{I}_L^N$,

- there exists a unique $\mathfrak{t}' \in \mathfrak{I}_{N-L}^N$ such that the images of $\mathfrak{t}$ and $\mathfrak{t}'$ are disjoint

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- there is associated a canonical permutation $\iota_{\mathfrak{t}} \in S_N$.
- we define $\mathrm{sgn}(\mathfrak{t}) := \mathrm{sgn}(\iota_{\mathfrak{t}})$.

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Given $\mathfrak{t},\mathfrak{u} \in \mathfrak{I}_L^N$, we define $W_{\mathfrak{t},\mathfrak{u}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ to be the $L \times L$ minor where

$$W_{\mathfrak{t},\mathfrak{u}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}[j,k] := W^{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathfrak{t}(j),\mathfrak{u}(k)].$$

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Given $\mathfrak{t},\mathfrak{u} \in \mathfrak{I}_L^N$, we define $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t},\mathfrak{u}}$ to be the $L \times L$ minor where

$$W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t},\mathfrak{u}}[j,k] := W^{\boldsymbol{\alpha},\boldsymbol{\beta}}[\mathfrak{t}(j),\mathfrak{u}(k)].$$

The complimentary minor is given by $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{u}'}$.

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We wish to enumerate the minors of $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

$$\mathfrak{I}_L^N := \left\{ \{1, 2, \ldots, L\} \xrightarrow{\mathfrak{t}} \{1, 2, \ldots, N\} \quad : \quad \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(L) \right\}.$$

For fixed $\mathfrak{u} \in \mathfrak{I}_L^N$, the Laplace expansion of $\det W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is given by

$$\det W^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \operatorname{sgn}(\mathfrak{u}) \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \det W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t},\mathfrak{u}} \cdot \det W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{u}'} .$$

# The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

$$\mathfrak{I}_L^N := \left\{\{1,2,\ldots,L\} \xrightarrow{\mathfrak{t}} \{1,2,\ldots,N\} \ : \ \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(L)\right\}.$$

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Let $\mathfrak{i} \in \mathfrak{I}_L^N$ be the identity map. Then, for every $\mathfrak{t} \in \mathfrak{I}_L^N$, the entries of $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}_{\mathfrak{t},\mathfrak{i}}$ do not depend on $\boldsymbol{\beta}$.

# *The Laplace Expansion of the Determinant*

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$$\det W^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \operatorname{sgn}(\mathfrak{u}) \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \det W_{\mathfrak{t},\mathfrak{u}}^{\boldsymbol{\alpha},\boldsymbol{\beta}} \cdot \det W_{\mathfrak{t}',\mathfrak{u}'}^{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

Let $\mathfrak{i} \in \mathfrak{I}_L^N$ be the identity map. Then, for every $\mathfrak{t} \in \mathfrak{I}_L^N$, the entries of $W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ do not depend on $\boldsymbol{\beta}$. Similarly, the entries of $W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\alpha},\boldsymbol{\beta}}$ do not depend on $\boldsymbol{\alpha}$.

# The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\boldsymbol{\alpha},\boldsymbol{\beta}}$.

$\mathfrak{I}_L^N := \big\{\{1,2,\ldots,L\} \xrightarrow{\mathfrak{t}} \{1,2,\ldots,N\} \ : \ \mathfrak{t}(1) < \mathfrak{t}(2) < \cdots < \mathfrak{t}(L)\big\}.$

Thus,

$$\det W^{\boldsymbol{\alpha},\boldsymbol{\beta}} = \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \det W^{\boldsymbol{\alpha}}_{\mathfrak{t},\mathfrak{i}} \cdot \det W^{\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{i}'}.$$

We have,

$$|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| = (-i)^M \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \operatorname{sgn} \Im(\beta_m) \right\} \det W^{\boldsymbol{\alpha},\boldsymbol{\beta}}.$$

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We have,

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We have,

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$$\times \left\{ \det W^{\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{i}'} (-i)^M \prod_{m=1}^{M} \operatorname{sgn} \Im(\beta_m) \right\}.$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \right\} \times |\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})| \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}),$$

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$$\times \left|\Delta(\boldsymbol{\alpha}, \boldsymbol{\beta})\right| d\lambda_L(\boldsymbol{\alpha})\, d\lambda_{2M}(\boldsymbol{\beta}),$$

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$$\times \left( \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \left\{ \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \right\} \right.$$

$$\left. \times \left\{ \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} (-i)^M \prod_{m=1}^{M} \operatorname{sgn} \Im(\beta_m) \right\} \right) d\lambda_L(\boldsymbol{\alpha})\, d\lambda_{2M}(\boldsymbol{\beta}),$$

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$$\times \left\{ \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \right\}$$
$$\times \left\{ \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} (-i)^M \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\}$$
$$\times\, d\lambda_L(\boldsymbol{\alpha})\, d\lambda_{2M}(\boldsymbol{\beta}),$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \frac{1}{L!M!} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M}$$

$$\times \left\{ \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \right\}$$

$$\times \left\{ \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} (-i)^M \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\}$$

$$\times \, d\lambda_L(\boldsymbol{\alpha}) \, d\lambda_{2M}(\boldsymbol{\beta}),$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \right\} d\lambda_L(\boldsymbol{\alpha})$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

# *The Idea of the Proof*

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \left\{ \prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) \right\} d\lambda_L(\boldsymbol{\alpha})$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

It is well known (De Bruijn 1955), that

$$\prod_{j<k} \operatorname{sgn}(\alpha_k - \alpha_j) = \operatorname{Pf} T^{\boldsymbol{\alpha}} \qquad \text{where} \qquad T^{\boldsymbol{\alpha}}[j,k] = \operatorname{sgn}(\alpha_k - \alpha_j).$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \operatorname{Pf} T^{\boldsymbol{\alpha}} \, d\lambda_L(\boldsymbol{\alpha})$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

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## The Idea of the Proof

$$\langle\Psi\rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \operatorname{Pf} T^{\boldsymbol{\alpha}} \, d\lambda_L(\boldsymbol{\alpha})$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

## *The Idea of the Proof*

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{\mathfrak{t},\mathfrak{i}}^{\boldsymbol{\alpha}} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \operatorname{Pf} T^{\boldsymbol{\alpha}} \, d\lambda_L(\boldsymbol{\alpha})$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

$$* = \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\}.$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right.$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W^{\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{i}'} \left\{ \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right.$$

$$\times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\mathfrak{t}',\mathfrak{i}'}^{\boldsymbol{\beta}} \left\{ \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}).$$

# The Idea of the Proof

$$\langle\Psi\rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau\in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right.$$

$$\left. \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W^{\boldsymbol{\beta}}_{\mathfrak{t}',\mathfrak{i}'} \left\{ \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \operatorname{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\boldsymbol{\beta}). \right.$$

$$* = \frac{1}{2^M M!} \sum_{\sigma\in S_{2M}} \operatorname{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(\mathfrak{t}'\circ\sigma)(2m-1)}, P_{(\mathfrak{t}'\circ\sigma)(2m)} \rangle_{\mathbb{C}}.$$

# *The Idea of the Proof*

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right.$$

$$\left. \times \left\{ \frac{1}{2^M M!} \sum_{\sigma \in S_{2M}} \operatorname{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(\mathfrak{t}'\circ\sigma)(2m-1)}, P_{(\mathfrak{t}'\circ\sigma)(2m)} \rangle_{\mathbb{C}} \right\} \right. .$$

# The Idea of the Proof

$$\langle\Psi\rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t}\in\mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau\in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)}\rangle_{\mathbb{R}} \right\}\right.$$

$$\times \left\{ \frac{1}{2^M M!} \sum_{\sigma\in S_{2M}} \operatorname{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(\mathfrak{t}'\circ\sigma)(2m-1)}, P_{(\mathfrak{t}'\circ\sigma)(2m)}\rangle_{\mathbb{C}} \right\}.$$

Define the $N \times N$ matrices $R$ and $C$ by

$$R[j,k] = \langle P_j, P_k\rangle_{\mathbb{R}} \quad \text{and} \quad C[j,k] = \langle P_j, P_k\rangle_{\mathbb{C}}.$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t})$$

$$\times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(\mathfrak{t}\circ\tau)(2\ell-1)}, P_{(\mathfrak{t}\circ\tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right.$$

$$\times \left\{ \frac{1}{2^M M!} \sum_{\sigma \in S_{2M}} \operatorname{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(\mathfrak{t}'\circ\sigma)(2m-1)}, P_{(\mathfrak{t}'\circ\sigma)(2m)} \rangle_{\mathbb{C}} \right\}.$$

These are (resp.) the Pfaffians of the minors $R_{\mathfrak{t},\mathfrak{t}}$ and $C_{\mathfrak{t}',\mathfrak{t}'}$.

# The Idea of the Proof

$$\langle \Psi \rangle = \mathfrak{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf} R_{\mathfrak{t},\mathfrak{t}} \cdot \operatorname{Pf} C_{\mathfrak{t}',\mathfrak{t}'}.$$

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf} R_{\mathfrak{t},\mathfrak{t}} \cdot \operatorname{Pf} C_{\mathfrak{t}',\mathfrak{t}'}.$$

It can be verified that

$$\sum_{(L,M)} \sum_{\mathfrak{t} \in \mathfrak{I}_L^N} \operatorname{sgn}(\mathfrak{t}) \operatorname{Pf} R_{\mathfrak{t},\mathfrak{t}} \cdot \operatorname{Pf} C_{\mathfrak{t}',\mathfrak{t}'} = \operatorname{Pf}(R + C).$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathcal{C}_N^{-1}\, \mathrm{Pf}(R + C).$$

# The Idea of the Proof

$$\langle \Psi \rangle = \mathcal{C}_N^{-1}\,\mathrm{Pf}(R + C).$$

$$R + C = U_{\mathbf{P}}.$$

Finally,

$$\langle \Psi \rangle = \mathcal{C}_N^{-1}\,\mathrm{Pf}(U_{\mathbf{P}}).$$

# *Averages over GinOE*

**Theorem 9 (S- 2006).** *Let $N = 2J$, and let*

$$\mathbf{P} = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\}$$

*be a set of monic polynomials with $\deg P_n = n - 1$. Then,*

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \, \mathrm{Pf}\, U_{\mathbf{P}},$$

*the Pfaffian of $U_{\mathbf{P}}$, where $U_{\mathbf{P}}$ is the $N \times N$ matrix where,*

$$U_{\mathbf{P}}[j,k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}},$$

*where $\langle P, Q \rangle_{\mathbb{R}}$ and $\langle P, Q \rangle_{\mathbb{C}}$ are defined with respect to*

$$\varphi(\gamma) = \psi(\gamma)\, e^{-\gamma^2/2} \left\{ \mathrm{erfc}(\sqrt{2}|\mathrm{Im}(\gamma)|) \right\}^{1/2}.$$

**Theorem 10 (S- 2005).** *Let $N = 2J$, and let*

$$\mathbf{P} = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\}$$

*be a set of monic polynomials with $\deg P_n = n - 1$. Then,*

$$F_N(\Phi; s) = \operatorname{Pf} U_{\mathbf{P}},$$

*the Pfaffian of $U_{\mathbf{P}}$, where $U_{\mathbf{P}}$ is the $N \times N$ matrix where,*

$$U_{\mathbf{P}}[j,k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}},$$

*where $\langle P, Q \rangle_{\mathbb{R}}$ and $\langle P, Q \rangle_{\mathbb{C}}$ are defined with respect to*

$$\varphi(\gamma) = \phi(\gamma)^{-s}.$$