

Review Sheet for Midterm 3

1. Evaluate

$$\iint_R e^{\frac{y-x}{y+x}} dA$$

where R is the triangle bounded by the line $x + y = 2$ and the coordinates axes.

2. Find $\iint_R \sqrt{x+y} dA$, where R is the parallelogram bounded by the lines

$$x + y = 0, \quad x + y = 1, \quad 2x - y = 0, \quad 2x - y = -3$$

3. Find the area of the region bounded by the curves

$$y = x^4, \quad y = 2x^4, \quad xy = 1, \quad xy = 6$$

4. For the following integrals

(a) sketch the region of integration.

(b) reverse the order of integration and evaluate it.

i. $\int_0^2 \int_{\frac{y}{2}}^1 ye^{x^3} dx dy$

ii. $\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (6x + 2y^2) dy dx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (6x + 2y^2) dy dx$

5. Find the volume of the solid that lies above the xy -plane, below $z = x$, and within the triangle whose vertices are $(1, 0)$, $(0, 2)$, and $(1, 2)$.
6. Compute the area of the region enclosed by the rose $r = \sin 2\theta$, given in polar coordinates.
7. Find the volume of the solid below the cone $z = \sqrt{x^2 + y^2}$, inside the cylinder $x^2 + y^2 = 2y$, and above the plane $z = 0$.

8. Compute

$$\iint_R \sqrt{9 - x^2 - y^2} dA,$$

where R is the region in the first quadrant within the circle $x^2 + y^2 = 9$.

9. Consider the parametric surface

$$x = u^2, \quad y = v^2, \quad z = u + v, \quad -\infty < u, v < \infty.$$

Find an equation of the tangent plane to the surface at the point $(1, 4, 3)$.

10. Find the area of the portion of the plane $2x + 2y + z = 8$ in the first octant.
11. Find the area of the portion of the cone $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + u \mathbf{k}$, for which $0 \leq u \leq 2v$, and $0 \leq v \leq \frac{\pi}{2}$.
12. Find the volume of the solid bounded by $z = 2 - x^2$, $z = x^2$, $y = 0$, and $y = 3$.

13. Find the volume of the tetrahedron bounded by $2x + 3y + z = 6$, $x = 0$, $y = 0$, and $z = 0$.
14. Express the following integral as an equivalent integral with z -integration first, y -integration next, and x -integration last.

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} f(x, y, z) \, dx \, dy \, dz$$

15. Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.
16. Use cylindrical or spherical coordinates to evaluate

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x \, dz \, dy \, dx$$

17. Use cylindrical or spherical coordinates to evaluate

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} dz \, dy \, dx$$

18. Evaluate $\int_C xz \, ds$ along

$$C : \mathbf{r}(t) = 6t \mathbf{i} + 3\sqrt{2}t^2 \mathbf{j} + 2t^3 \mathbf{k}, \quad 0 \leq t \leq 1.$$

19. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (y + z) \mathbf{i} - x^2 \mathbf{j} - 4y^2 \mathbf{k}$, and

$$C : \mathbf{r}(t) = t \mathbf{i} + t^2 \mathbf{j} + t^4 \mathbf{k}, \quad 0 \leq t \leq 1.$$

20. Find the work done by the force field $\mathbf{F}(x, y) = x^2 \mathbf{i} + xy \mathbf{j}$ on a particle that moves around the circle $x^2 + y^2 = 4$ in the counter clockwise direction.

21. Given that $\mathbf{F}(x, y) = 2xy^3 \mathbf{i} + 3x^2y^2 \mathbf{j}$ and

$$C : \mathbf{r}(t) = \sin t \mathbf{i} + (t^2 + 1) \mathbf{j}, \quad 0 \leq t \leq \frac{\pi}{2},$$

- (a) find the potential function for \mathbf{F} .
- (b) find the work done moving a particle along C in the force field \mathbf{F} .

1. Answer: $\boxed{\iint_R e^{\frac{y-x}{y+x}} dA = e - \frac{1}{e}}.$

2. Answer: $\boxed{\iint_R \sqrt{x+y} dA = \frac{2}{3}}.$

3. Answer: $\boxed{\ln 2}.$

4. Answer:

i. (a) The region is a triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$.

(b)

$$\int_0^2 \int_{\frac{y}{2}}^1 ye^{x^3} dx dy = \int_0^1 \int_0^{2x} ye^{x^3} dy dx = \dots = \boxed{\frac{2}{3}(e-1)}$$

ii. (a) The region is bounded by $x = y^2$ and $x = 2 - y$. Note that these curves intersect at $(1, 1)$ and $(4, -2)$.

(b)

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} (6x+2y^2) dy dx + \int_1^4 \int_{-\sqrt{x}}^{2-x} (6x+2y^2) dy dx = \int_{-2}^1 \int_{y^2}^{2-y} (6x+2y^2) dx dy = \dots = \boxed{\frac{99}{2}}$$

5. Answer: $V = \int_0^1 \int_{2-2x}^2 x dy dx = \dots = \boxed{\frac{2}{3}}$

6. Answer: $A = 4 \int_0^{\frac{\pi}{2}} \int_0^{\sin 2\theta} r dr d\theta = \dots = \boxed{\frac{\pi}{2}}$. Note: $\sin^2 2\theta = \frac{1}{2}(1 - \cos 4\theta)$

7. Answer: $V = \int_0^\pi \int_0^{2\sin\theta} \int_0^r r dz dr d\theta = \dots = \boxed{\frac{32}{9}}$. Note: $\sin^3 \theta = (1 - \cos^2 \theta) \sin \theta$

8. Answer: Use polar coordinates.

$$\iint_R \sqrt{9-x^2-y^2} dA = \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{9-r^2} r dr d\theta = \dots = \boxed{\frac{9\pi}{2}}$$

9. Answer: The point $(1, 4, 3)$ is obtained when $u = 1$ and $v = 2$. Then

$$\mathbf{w} = \frac{\partial \mathbf{r}}{\partial u}(1, 2) \times \frac{\partial \mathbf{r}}{\partial v}(1, 2)$$

is normal to the surface, where $\mathbf{r}(u, v) = u^2 \mathbf{i} + v^2 \mathbf{j} + (u+v) \mathbf{k}$. We compute

$$\frac{\partial \mathbf{r}}{\partial u} = 2u \mathbf{i} + \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = 2v \mathbf{j} + \mathbf{k}$$

so $\frac{\partial \mathbf{r}}{\partial u}(1, 2) = 2 \mathbf{i} + \mathbf{k}$ and $\frac{\partial \mathbf{r}}{\partial v}(1, 2) = u \mathbf{j} + \mathbf{k}$. Thus, $\mathbf{w} = -4 \mathbf{i} - 2 \mathbf{j} + 8 \mathbf{k}$. Therefore, the equation of the tangent plane is $-4(x-1) - 2(y-4) + 8(z-3) = 0$, which simplifies to

$$\boxed{2x + y - 4z = -6}.$$

10. **Answer:** We have $z = 8 - 2x - 2y$ so $\frac{\partial z}{\partial x} = -2$ and $\frac{\partial z}{\partial y} = -2$. Thus

$$A = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} dA = 3 \iint_R dA = \boxed{24}$$

where R is the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 4)$. Note: R is a triangle with base and height equal to 4, so $\iint_R dA = 8$.

11. **Answer:** We have

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v \mathbf{i} + \sin v \mathbf{j} + \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u \sin v \mathbf{i} + u \cos v \mathbf{j}.$$

Thus,

$$\left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| = \left\| -u \cos v \mathbf{i} - u \sin v \mathbf{j} + u \mathbf{k} \right\| = \sqrt{2u^2}.$$

Now,

$$A = \iint_R \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\| du dv = \int_0^{\frac{\pi}{2}} \int_0^{2v} \sqrt{2u^2} du dv = \dots = \boxed{\frac{\sqrt{2}\pi^3}{12}}.$$

12. **Answer:** $V = \int_0^3 \int_{-1}^1 \int_{x^2}^{2-x^2} dz dx dy = \dots = \boxed{8}$.

13. **Answer:** $V = \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} dz dy dx = \dots = \boxed{6}$.

14. **Answer:** This is the 1st octant part of the sphere $x^2 + y^2 + z^2 = 4$, so

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_0^{\sqrt{4-y^2-z^2}} f(x, y, z) dx dy dz = \boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} f(x, y, z) dz dy dx}.$$

15. **Answer:** $V = 2 \int_0^{2\pi} \int_0^1 \int_0^{\sqrt{4-r^2}} r dz dr d\theta = \dots = \boxed{\frac{32\pi}{3} - 4\pi\sqrt{3}}$.

16. **Answer:**

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} x dz dy dx = \int_0^{\frac{\pi}{2}} \int_0^3 \int_0^{9-r^2} r^2 \cos \theta dz dr d\theta = \dots = \boxed{\frac{162}{5}}.$$

17. **Answer:**

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{-5+x^2+y^2}^{3-x^2-y^2} dz dy dx = \int_0^{\frac{\pi}{2}} \int_0^2 \int_{r^2-5}^{3-r^2} r dz dr d\theta = \dots = \boxed{4\pi}.$$

18. **Answer:**

$$\int_C xz ds = \int_0^1 (6t)(2t^3) \left\| 6\mathbf{i} + 6\sqrt{2}t\mathbf{j} + 6t^2\mathbf{k} \right\| dt = \dots = 72 \int_0^1 t^4(t^2 + 1) dt = \dots$$

19. Answer:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [(t^2 + t^4)\mathbf{i} - t^2\mathbf{j} - 4t^4\mathbf{k}] \cdot [\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}] dt = \int_0^1 (t^2 + t^4 - 2t^3 - 16t^7) dt = \dots$$

20. Answer:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (4\cos^2 t\mathbf{i} + 4\cos t \sin t\mathbf{j}) \cdot (-2\sin t\mathbf{i} + 2\cos t\mathbf{j}) dt = \dots$$

21. Answer:

(a) Since $\frac{\partial}{\partial y}(2xy^3) = 6xy^2$ and $\frac{\partial}{\partial x}(3x^2y^2) = 6xy^2$, \mathbf{F} is conservative. If we let f be the potential function for \mathbf{F} , we have $\nabla f = \mathbf{F}$, so $\frac{\partial f}{\partial x} = 2xy^3$ and $\frac{\partial f}{\partial y} = 3x^2y^2$. Integrate $\frac{\partial f}{\partial x}$ with respect to x to find that $f(x, y) = x^2y^3 + g(y)$. Since $\frac{\partial f}{\partial y} = 3x^2y^2$, we find that $g'(y) = 0$, so $g(y)$ is just a constant K . Thus f is of the form $f(x, y) = x^2y^3 + K$. Any value for K will satisfy the equation $\nabla f = \mathbf{F}$, so we might as well choose $K = 0$. Thus $\boxed{f(x, y) = x^2y^3}$ is a potential function for \mathbf{F} (of course $f(x, y) = x^2y^3 + K$ is as well for any constant K).

(b) $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = \boxed{f(1, \frac{\pi^2}{4} + 1) - f(0, 1)}$.