Pacific Northwest Number Theory Conference

Conjugate Reciprocal Polynomials with all Roots on the Unit Circle

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Acknowledgments

Joint work with Kathleen Petersen (Queen's University).

Definitions

A polynomial $f \in \mathbb{C}[x]$ is *conjugate reciprocal* (CR) if

$$f(\overline{x}) = x^N \overline{f(1/\overline{x})}, \qquad N = \deg f.$$

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- *reciprocal:* $c_{N-n} = c_n$
- *self-inversive:* $c_{N-n} = \xi\,\overline{c_n}, \quad$ for some $\quad |\xi| = 1$.

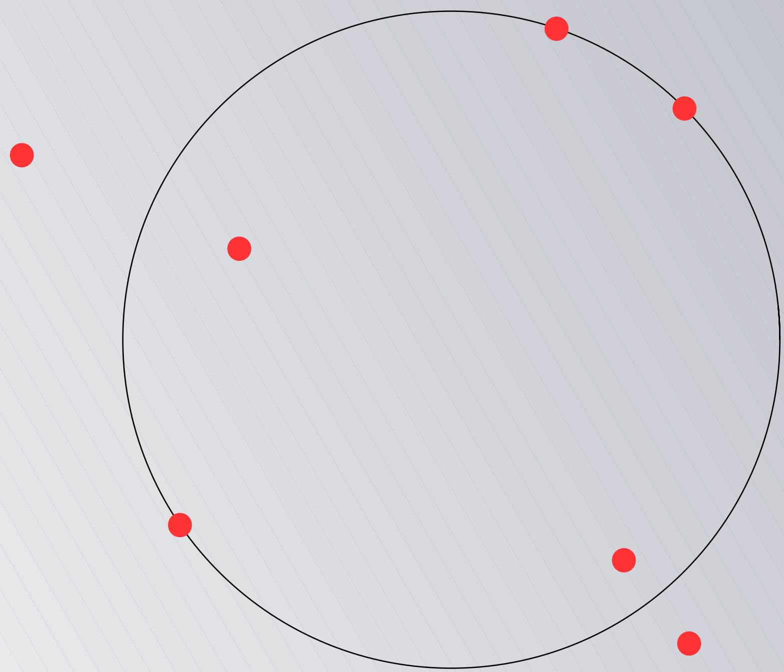
The CR condition implies

$$f(\alpha) = 0 \qquad \Rightarrow f(1/\overline{\alpha}) = 0$$

Moreover $\alpha = 1/\overline{\alpha}$ if and only if $\alpha \in \mathbb{T}$.

Thus the roots of f are either on $\mathbb{T}$ or come is pairs invariant under inversion across $\mathbb{T}$ (hence the term self-inversive).

Definitions



A plot of the roots of

$$x^7 + (\tfrac{1}{2} - \tfrac{i}{3})x^6 - (\tfrac{1}{3} - \tfrac{i}{2})x^5 + \tfrac{3}{2}x^4 + \tfrac{3}{2}x^3 - (\tfrac{1}{3} + \tfrac{i}{2})x^2 + (\tfrac{1}{2} + \tfrac{i}{3})x + 1$$

More Definitions

We will be primarily interested in monic CR polynomials. That is, polynomials of the form

$$f(x) = x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n} \qquad c_{N-n} = \overline{c_n}$$

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The number of real numbers needed to describe f is

$$N - 1 = \begin{cases} 2\left(\frac{N-1}{2}\right) & N \text{ odd} \\ 2\left(\frac{N-2}{2}\right) + 1 & N \text{ even.} \end{cases}$$

Yet More Definitions

We may identify monic CR polynomials of degree N with $\mathbb{R}^{N-1}$. To do this define the $N-1 \times N-1$ matrix X_N by

$$X_N[j,k] = \begin{cases} \frac{\sqrt{2}}{2}\left(\delta_{j,k} + \delta_{N-j,k}\right) & \text{if} \quad 1 \le j < \lfloor N/2 \rfloor \\ \delta_{j,k} & \text{if} \quad j = N/2 \\ \frac{\sqrt{2}}{2}\left(i\delta_{N-j,k} - i\delta_{j,k}\right) & \text{if} \quad \lfloor N/2 \rfloor < j < N \end{cases}$$

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$$X_5 = \frac{\sqrt{2}}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -i \end{bmatrix}$$

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Notice if $\mathbf{a} \in \mathbb{R}^{N-1}$ then $X_N \mathbf{a} =: \mathbf{c} \in \mathbb{C}^{N-1}$ and $c_{N-n} = \overline{c_n}$.

Yet More Definitions

Since $\mathbf{c} = X_N \mathbf{a}$ has the property

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$$\mathbf{a}(x) := x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

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Since $\mathbf{c} = X_N \mathbf{a}$ has the property

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we may use this to define a monic CR polynomial.

$$\mathbf{a}(x) := x^N + 1 + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

We explicitly identify the set of monic CR polynomials of degree N with $\mathbb{R}^{N-1}$ by the map

$$\mathbf{a} \mapsto \mathbf{a}(x).$$

The Definition of W_N

Finally we define,

$$W_N = \{\mathbf{a} \in \mathbb{R}^{N-1} : \mathbf{a}(x) \text{ has all roots on } \mathbb{T}\}.$$

That is, W_N is the collection of vectors in $\mathbb{R}^{N-1}$ which correspond to degree N CR polynomials with all roots on the unit circle.

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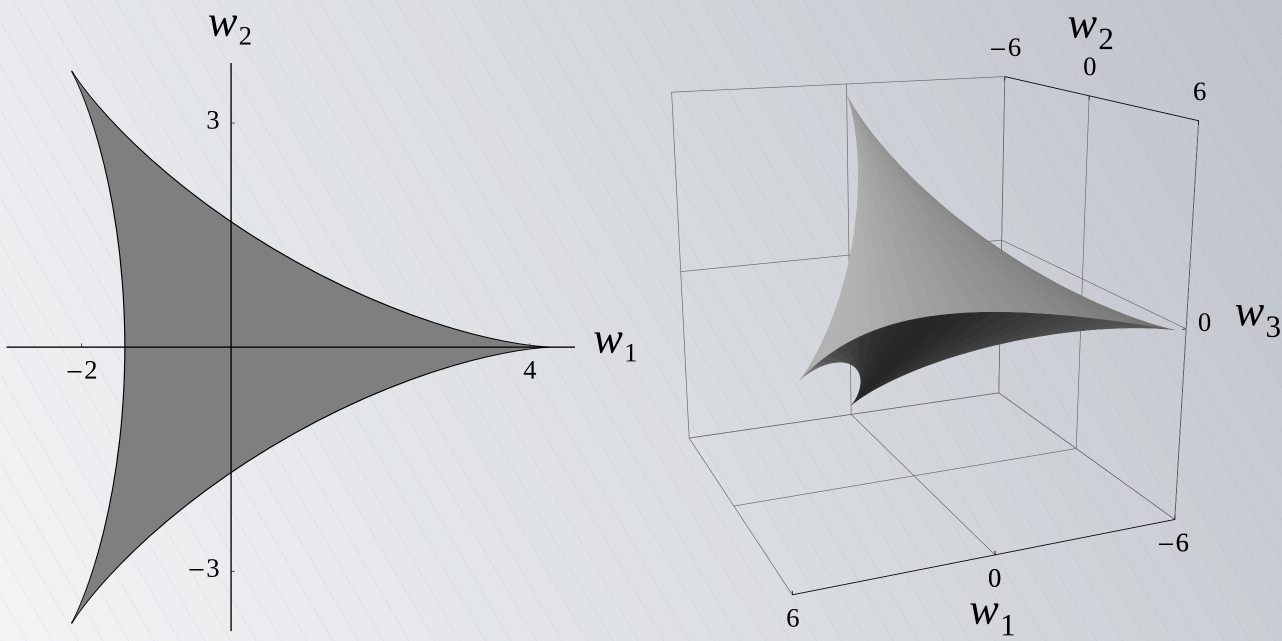

Plots of W_3 and W_4.

Theorems

Theorem 1. W_N *is homeomorphic to the closed* $N-1$ *dimensional ball,* B^{N-1}*.*

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Given $\mathbf{a} \in W_N$ there is a natural way to define a (cyclically ordered) partition of N, $\mathcal{P}(\mathbf{a})$ corresponding to the multiplicities of the cyclically ordered roots of $\mathbf{a}$

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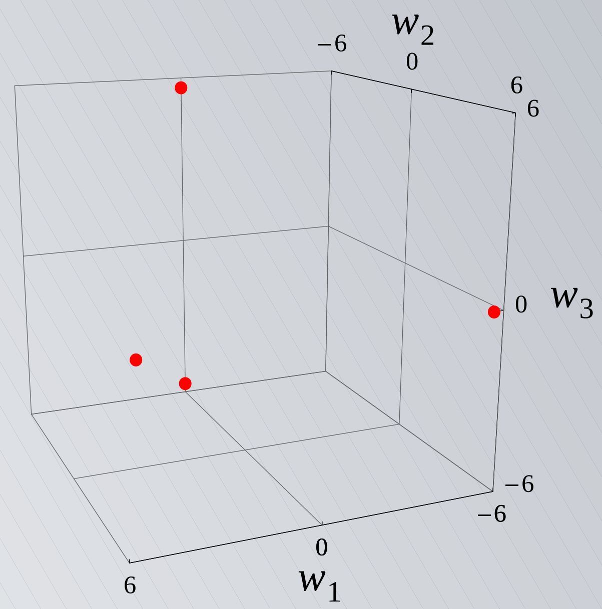
Points in W_4 corresponding to the partition $\{4\}$.



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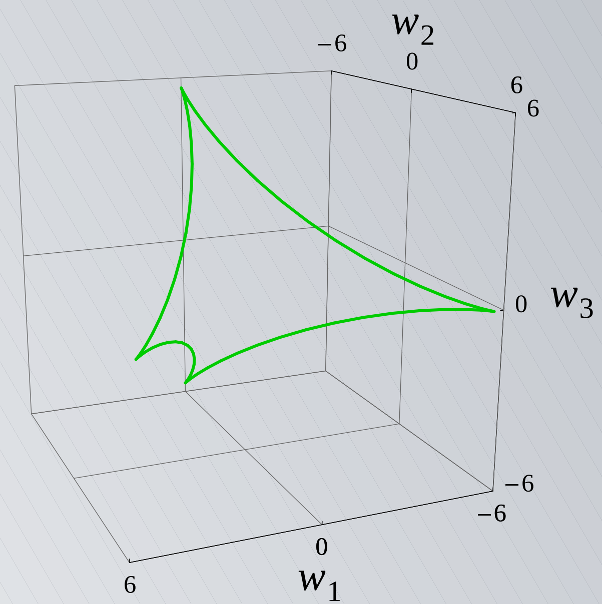
Points in W_4 corresponding to the partition $\{3,1\}$.



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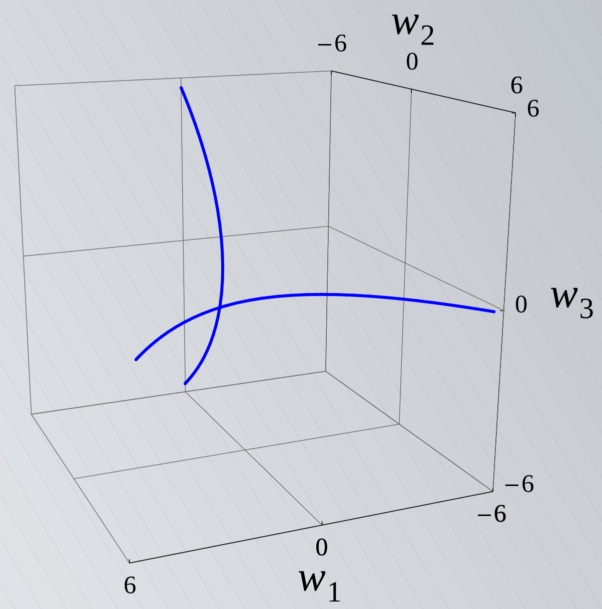
Points in W_4 corresponding to the partition $\{2, 2\}$.



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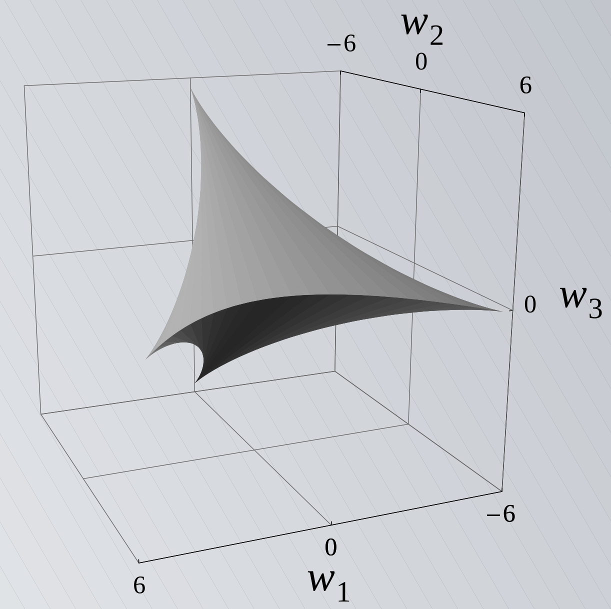
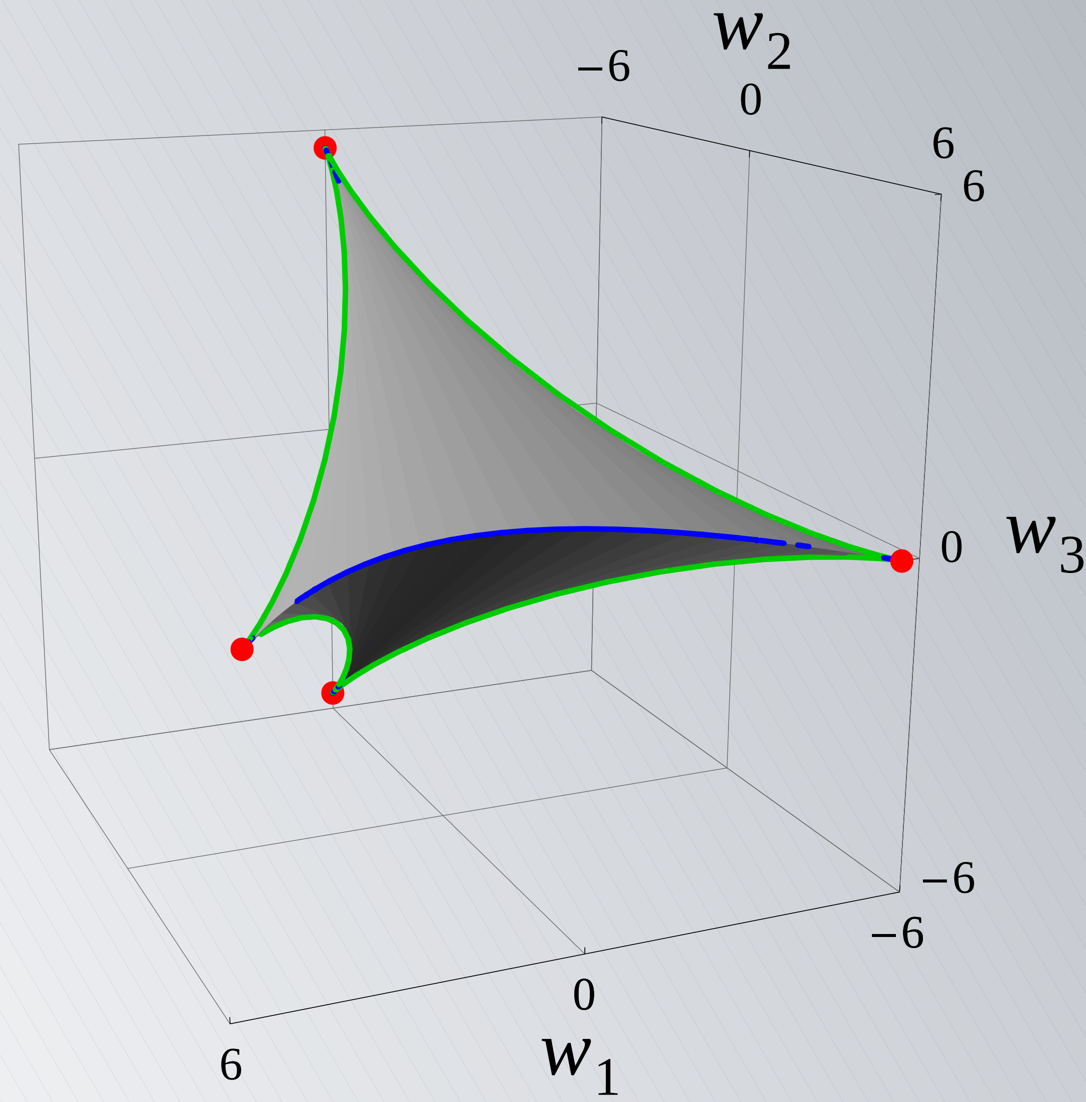
Points in W_4 corresponding to the partition $\{1, 1, 1, 1\}$.



W_4 *Coloured*

$\{{\color{red}4}\}$,
$\{{\color{green}3, 1}\}$,
$\{{\color{blue}2, 2}\}$,
$\{1, 1, 1, 1\}$,



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Theorem 3. *The group of isometries of* W_N *is isomorphic to* D_N*, the dihedral group of order* $2N$*.*

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$\mathrm{Isom}(W_N)$ is generated by the isometries R and C where R corresponds to the action of multiplying each root by a primitive N-th root of unity, and C corresponds to complex conjugation of the roots.

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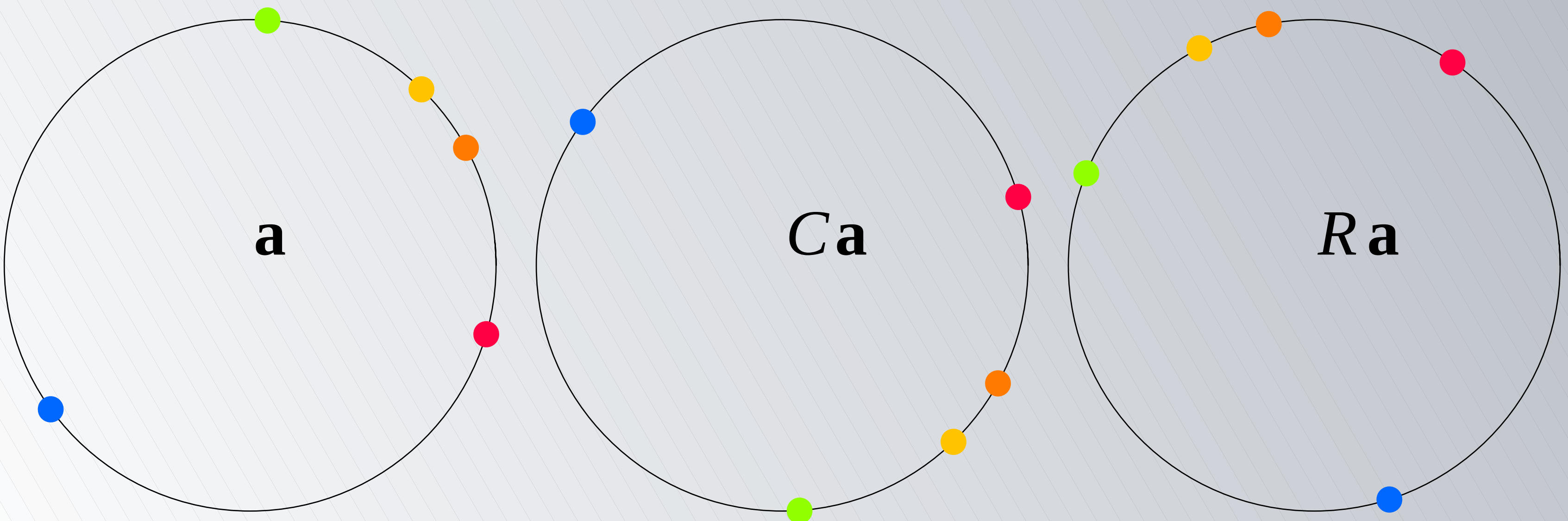

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$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$

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$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma(\frac{N+1}{2})}.$$

That is, the volume of W_N is equal to the volume of the $N-1$ dimensional ball of radius 2.

Proposition 5. *The vector* $\mathbf{w}$ *is in* W_N *if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^{N} (x - \xi_n),$$

where $\xi_1, \xi_2, \ldots, \xi_N$ *are elements of* $\mathbb{T}$ *satisfying* $\xi_1 \xi_2 \cdots \xi_N = (-1)^N$*.*

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Proof.

$$x^N \overline{\mathbf{w}(1/\overline{x})} = x^N \prod_{n=1}^{N} \left(\frac{1 - \overline{\xi_n} x}{x} \right)$$

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It follows that $\mathbf{w}$ is CR and in W_N. The converse is obvious since every element of W_N is a polynomial with all roots on the unit circle and constant coefficient 1.

Proposition 6. *The points in W_N corresponding to the partition $\{N\}$ are given by*

$$\mathbf{v}_n(x) = (x + \zeta_N^n)^N \qquad n = 1, 2, \ldots, N,$$

where $\zeta_N = e^{2\pi i/N}$.

A Few More Preliminaries

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where $\zeta_N = e^{2\pi i/N}$. Moreover, if $\mathbf{w} \in W_N$ then

$$\|\mathbf{w}\|^2 \le \binom{2N}{N} - 2,$$

with equality if and only if $\mathbf{w} = \mathbf{v}_n$ for some $1 \le n \le N$.

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We call the $\mathbf{v}_n$ the *vertices* of W_N.

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with equality if and only if $\mathbf{w} = \mathbf{v}_n$ *for some* $1 \leq n \leq N$.

Corollary 7. *Each Isometry of* W_N *must map the set of vertices to itself. That is,* $\mathrm{Isom}(W_N)$ *is a subgroup of* S_N.

Isometries and Parseval's Formula

To show R is a an isometry of W_N we need to show for any points $\mathbf{w}_1, \mathbf{w}_2 \in W_N$,

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The $N-1 \times N-1$ matrix X_N is normalized so that $\|X_N \mathbf{w}\|^2 = \|\mathbf{w}\|^2$. It follows that

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Now, since $R\mathbf{w}(x) = \mathbf{w}(\zeta_N^{-1}x)$,

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Lemma 8. *For each* $1 \leq M \leq N$*, the vertices farthest from* $\mathbf{v}_M$ *are* $\mathbf{v}_{M+1}$ *and* $\mathbf{v}_{M-1}$*.*

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- T is in the group spanned by R and C.

Self Inversive Polynomials

The polynomial f is said to be *self-inversive* if there exists $\omega \in \mathbb{T}$ such that

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For each ω we will call such polynomials, ω-CR polynomials. Note that 1-CR polynomials are just CR polynomials.

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As in the CR case, the set of monic ω-CR polynomials is in bijective correspondence with $\mathbb{R}^{N-1}$.

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We define

$$W_{N,\omega} = \left\{ \mathbf{a} \in \mathbb{R}^{N-1} : (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n} \text{ has all roots on } \mathbb{T} \right\}$$

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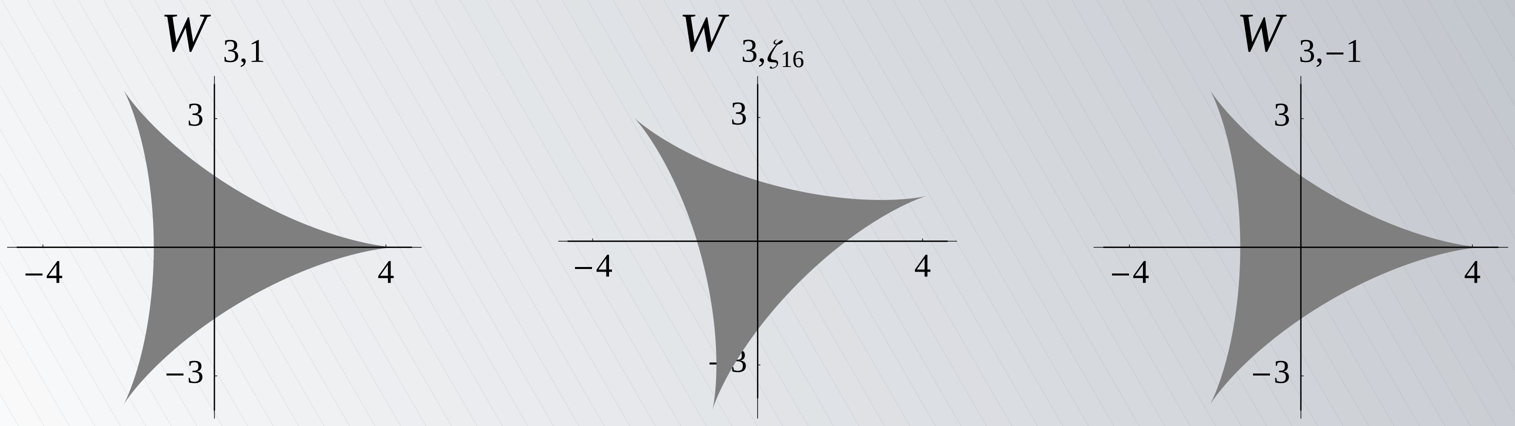


A Change of Variables

Proposition 8. *The vector* $\mathbf{w}$ *is in* $W_{N,\omega}$ *if and only if*

$$\mathbf{w}(x) = \prod_{n=1}^{N} (x - \xi_n),$$

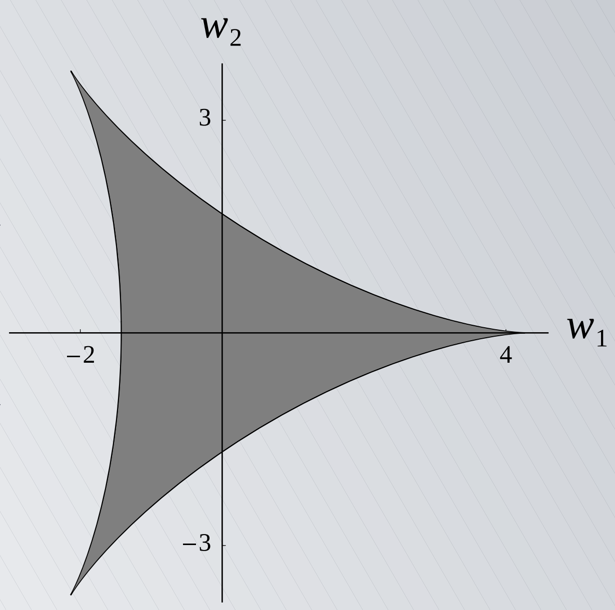
where $\xi_1, \xi_2, \ldots, \xi_N$ *are elements of* $\mathbb{T}$ *satisfying* $\xi_1 \xi_2 \cdots \xi_N = (-1)^N \omega$*.*

We define the map $E_{N,\omega} : \mathbb{T}^{N-1} \to W_{N,\omega}$ specified by $\mathbf{a} = E_{N,\omega}(\boldsymbol{\xi}) = X_{N,\omega}^{-1}\mathbf{c}$ where $\mathbf{c}$ is obtained from $\boldsymbol{\xi}$ by

$$\left(x - \frac{(-1)^N \omega}{\xi_1 \xi_2 \cdots \xi_{N-1}}\right) \prod_{n=1}^{N-1} (x - \xi_n) = (x^N + \omega) + \sum_{n=1}^{N-1} c_n x^{N-n}.$$

The Volume of W_N

$$\mathrm{vol}(W_N) = \mathrm{vol}\Bigg($$



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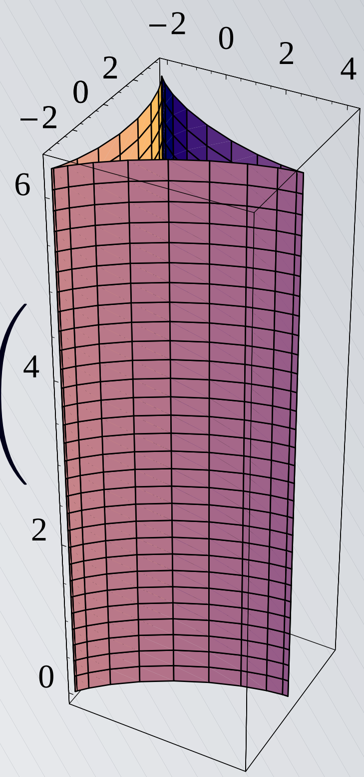

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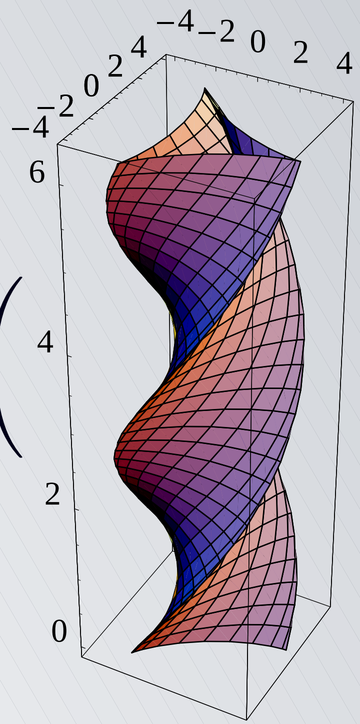

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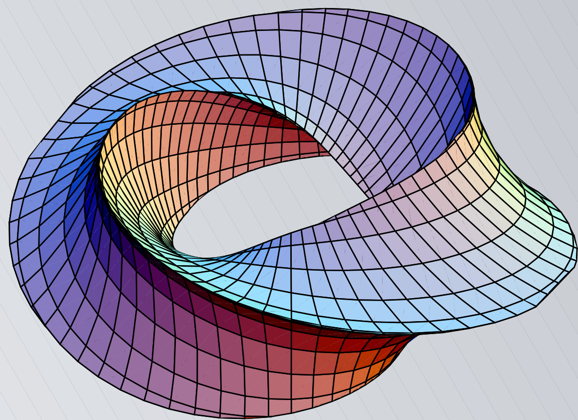

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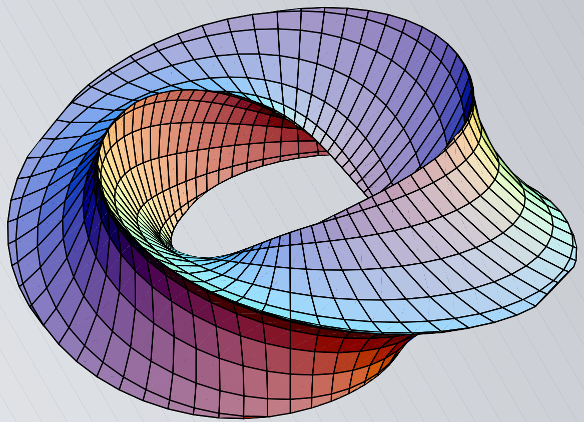
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg($$  $$\Bigg)$$

That is,

$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{vol}(W_{N,e^{i\theta}})\,d\theta$$

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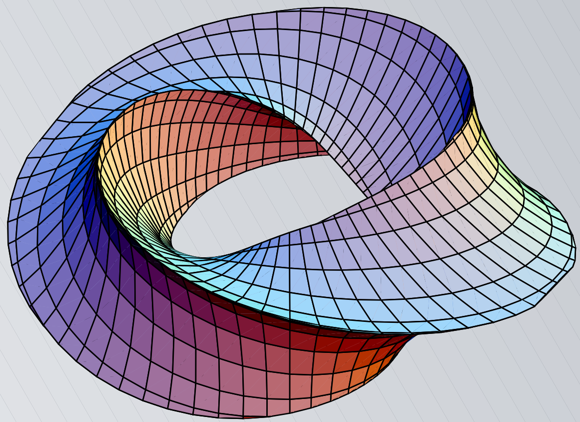
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\left(\quad\right.$$  $$\left.\quad\right)$$

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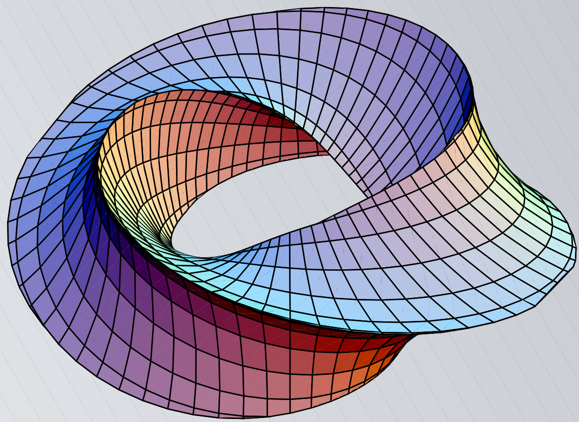
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg($$  $$\Bigg)$$

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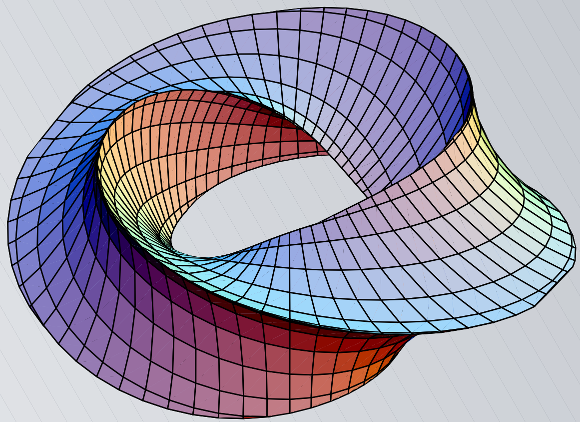
$$\mathrm{vol}(W_N) = \frac{1}{2\pi}\,\mathrm{vol}\Bigg($$  $$\Bigg)$$

Dyson computed this integral in the context of RMT, and

$$\mathrm{vol}(W_N) = \frac{2^{N-1}\pi^{(N-1)/2}}{\Gamma\left(\frac{N+1}{2}\right)}.$$