Introduction to Probability
HW9 Solutions

(6.48) There is a formula for the density of any order statistic. In our case it reads

$$f_{X^{(3)}}(x) = \frac{5!}{2!2!}x^2(1-x)^2$$

on $x \in [0,1]$. So you have to integrate:

$$\begin{aligned} P(1/4 < X^{(3)} < 3/4) &= 30\int_{1/4}^{3/4} x^2(1-x)^2dx \\ &= \left(10x^3 - 15x^4 + 6x^5\right)\Big|_{1/4}^{3/4} = \frac{203}{16\cdot 16}. \end{aligned}$$

(6.49)

(a) $P(MIN \le a) = 1 - [P(X_1 > a)]^5 = 1 - e^{-5\lambda a}$.

(b) $P(MAX \le a) = (1 - e^{-\lambda a})^5$.

(6.51) $f(r,\theta) = 1_{\{0 \le r\cos\theta \le 1\}}1_{\{0 \le r\sin\theta \le 1\}}r$.

(6 TH 9) By the computation we did in class this has distribution $\exp(n\lambda)$.

(6 TH 11)

$$\begin{aligned} I &= \int\cdots\int_{-\infty<x_1<x_2<\cdots<x_5<\infty} dF(x_1)dF(x_2)\cdots dF(x_5) \\ &= \int_0^1\int_0^{u_1}\int_0^{u_2}\int_0^{u_3}\int_0^{u_4} du_5du_4du_3du_2du_1 = \frac{1}{5!} \end{aligned}$$

after the change of variables $u_k = F(u_k)$ for $k = 1,2,3,4,5$.

(6 TH 17)

(a)

$$\begin{aligned} P(X_1 > X_2 | X_1 > X_3) &= \frac{P(X_1 > X_2 > X_3) + P(X_1 > X_3 > X_2)}{P(X_1 > X_3)} \\ &= \frac{1/6 + 1/6}{1/2} = 2/3. \end{aligned}$$

(b)

$$\begin{aligned} P(X_1 > X_2 | X_1 < X_3) &= \frac{P(X_3 > X_1 > X_2)}{P(X_1 < X_3)} \\ &= \frac{1/6}{1/2} = 1/3. \end{aligned}$$

(7.3) Just an integral:

$$\begin{aligned} E[|X-Y|^\alpha] &= \int_0^1\int_0^1 |x-y|^\alpha dxdy \\ &= 2\int_0^1\int_y^1 (x-y)^\alpha dxdy \\ &= \frac{2}{(1+\alpha)}\int_0^1 (1-y)^{\alpha+1}dy = \frac{2}{(1+\alpha)}\int_0^1 z^{\alpha+1}dz = \frac{2}{(1+\alpha)(2+\alpha)}. \end{aligned}$$

(7.7) Let cA and cB denote "chosen by A" and "chosen by B". Then

(a) $E[\#\text{chosen by both}] = 10 \cdot P(\text{object 1 } cA \text{ and } cB) = 10 \cdot (3/10)^2 = .9$

(b) $E[\#\text{not chosen}] = 10 \cdot P(\text{object 1 neither } cA \text{ nor } cB) = 10 \cdot (9/10)^2 = 4.9$

(c) $E[\#\text{chosen by only one of A or B}] = 10 \cdot 2 \cdot (3/10) \cdot (7/10) = 4.2.$

Note that together these sum to 10 as they should.

(7.8) Let

$$\#\{\text{tables needed}\} = X_1 + X_2 + \cdots + X_N$$

where

$$X_k = 1_{\{\text{person k needs a new table}\}}.$$

Then,

$$\begin{aligned} E[\#\{\text{tables needed}\}] &= \sum_{k=1}^{N} P\Big(\text{person k knows nobody in the room}\Big) \\ &= \sum_{k=0}^{N-1} (1-p)^k = \frac{1}{p}\Big(1-(1-p)^N\Big) \end{aligned}$$

(7.18) Let N = number of matches. Then

$$N = M_1 + M_2 + \cdots + M_{52}$$

where M_k is the indicator of the event that there is a match at time k. The M_k are all equally distributed, so

$$E[N] = 52\, P(\text{match at time 1}) = 52 \cdot \frac{4}{52} = 4.$$

(7.22) This is just the coupon collecting problem done in class with 6 coupons. Repeating that analysis the expectation desired is $6(1 + 1/2 + 1/3 + \cdots + 1/6)$.

(7.34) If N is the number of couples sitting next to each other, number the wives say and let W_k be the event that wife k sits next to her spouse. Then

$$N = 1_{W_1} + 1_{W_2} + \cdots + 1_{W_{10}}$$

and

$$E[N] = 10P(W_1) = 10 \cdot (2/19).$$

(It is a round table!). In a similar fashion:

$$VAR[N] = 10\Big(P(W_1) - P^2(W_1)\Big) + 90\Big(P(W_1 \cap W_2) - P^2(W_1)\Big).$$

The only new thing to compute is

$$P(W_1 \cap W_2) = P(W_1 \,|\, W_2)P(W_2) = (\frac{16}{18}\frac{2}{17} + \frac{2}{18}\frac{1}{17})\frac{2}{19}.$$

From here a bit of algebra shows the variance is 360/361

(7TH10) Note that for each $k = 1, \dots n$

$$Y_k = \frac{X_k}{\sum_{j=1}^{n} X_j}$$

has the same distribution. So

$$E[Y_k] = constant$$

and obviously

$$E[Y_1 + Y_2 + \cdots + Y_n] = n \cdot constant = 1.$$

That is, $constant = 1/N$. It follows that

$$E\left[\frac{\sum_{j=1}^{k} X_j}{\sum_{j=1}^{n} X_j}\right] = \frac{k}{n}.$$

(8.1) If X has mean and variance 20,

$$P(0 < X < 40) = P(|X - 20| < 20) = 1 - P(|X - 20| > 20) \geq 1 - 20/(20)^2 = 19/20$$

by Chebychev.

(8.4(a)) Markov says

$$P(\sum_{1}^{20} X_k > 15) \leq 20E[X_1]/15 = 4/3$$

which isn't too informative!

(8.9) By Chebychev, if $X \sim \Gamma(n, 1)$:

$$P\Big(|\frac{X}{n} - 1| > .01\Big) \leq \frac{1}{(.01)^2 n}.$$

Now find n which makes the right hand side less than .01: $n \geq 10^6$.