Introduction to Probability
HW6 Solutions

(4 Th 20) The answer is (b). The procedure of not considering the Binomial unless it is greater than zero is the same as *conditioning* on that event.

(5.7) $f(x) = a + bx^2$ on $[0,1]$ and 0 otherwise. This gives two equations for the unknowns a and b:

$$1 = \int_0^1 (a + bx^2)dx = a + b/3$$

and

$$3/5 = \int_0^1 x(a + bx^2)dx = a/2 + b/4.$$

Now solve: $a = 3/5$, $b = 6/5$.

(5.9) X is demand with density $f(x)$ and $F(s) = \int_0^s f(x)dx$. The profit is:

$$P(s,X) = (bX - (s - X)\ell)1_{\{X<s\}} + sb1_{\{X>s\}}$$

and

$$\begin{aligned} E[P(s,X)] &= (b+\ell)\int_0^s xf(x)dx - s\ell\int_0^s f(x)dx + sb\int_s^\infty f(x)dx \\ &= (b+\ell)\int_0^s xf(x)dx - s(b+\ell)\int_0^s f(x)dx + sb. \end{aligned}$$

Setting derivative equal to zero you find:

$$0 = b - (b+\ell)\int_0^s f(x)dx$$

which determines a unique solution s^* as $F(s)$ is increasing from 0 to 1.

(5.11) $X \sim U[0,L]$. There are two lengths, X and $L - X$. We want

$$\begin{aligned} Prob &= P(X/(L-X) \le 1/4,\ X \le L/2) + P((L-X)/X \le 1/4,\ X > L/2) \\ &= 2P(X/(L-X) \le 1/4,\ X \le L/2) \\ &= 2P(X \le L/5) = 2/5. \end{aligned}$$

The obvious symmetry is used in line two.

(5.26) Normal Apprx.

$$\begin{aligned} P(\text{see } 525 \text{ heads} \mid \text{fair coin}) &= P(X_{1000,1/2} \ge 525) \\ &= P\Big(\frac{X - 500}{\sqrt{250}} \ge 25/\sqrt{250}\Big) \\ &\approx P(Z \ge 5/\sqrt{10}) \qquad \text{where } Z \sim N(0,1) \\ &= 1 - \Phi(5/\sqrt{10}). \end{aligned}$$

Part two is much the same.

(5.37) (a) Split into cases:

$$P(|X| > 1/2) = P(X < -1/2) + P(X > 1/2) = 1/4 + 1/4 = 1/2.$$

(b) Find distribution function:

$$F_{|X|}(x) = P(|X| \leq x) = \frac{1}{2}\int_{-x}^{x} dx = x.$$

That is, the density $f_{|X|}(x) = 1$ for $x \in [0,1]$ (and otherwise 0). That is, $|X|$ is $U[0,1]$.

(5.38) For roots to be real need

$$(4Y)^2 - 4 \cdot 4(Y+2) > 0$$

which is the same as

$$Y^2 - Y - 2 > 0.$$

Thus want $Y < -1$ or $Y > 2$. But Y is uniform on $[0,5]$, so really asking

$$P(2 < Y \leq 5) = 3/5.$$

(5 Th 5) Just change order of integration:

$$\begin{aligned}
\int_0^\infty nx^{n-1}P(X > x)dx &= \int_0^\infty \int_x^\infty nx^{n-1} f_X(z)dzdx \\
&= \int_0^\infty \int_0^z nx^{n-1} f_X(z)dxdz \\
&= \int_0^\infty z^n f_X(z)dz
\end{aligned}$$

and this is the definition of $E[Z]$.