Introduction to Probability
HW5 Solutions

(4.36) $E[X^2] = 4(p^2 + (1-p)^2) + 18(p(1-p)^2 + p^2(1-p))$ and have from before that $E[X] = 2(p^2 + (1-p)^2) + 6(p(1-p)^2 + p^2(1-p))$. Thus

$$\frac{d}{dp}Var[X] = 8(1-2p) + 18((1-p)^2 - p^2) - 2E[X](d/dp)E[X].$$

First two terms vanish when $p = 1/2$ and already know that last term vanishes there as well.

(4.38) (a)

$$\begin{aligned} E[(2+X)^2] &= E[X^2] + 4E[X] + 4 \\ &= Var[X] + (E[X])^2 + 4E[X] + 4 = 5 + 1 + 4 + 4 = 14. \end{aligned}$$

(b) Var[4 + 3 X] = 9 Var[X] = 45.

(4.51) Typos are rare events so this problem is prompting you to consider the number of such occurances as being well modeled as a Poisson of parameter .2 = mean number or typos. So,

$$P(0 \text{ errors }) = e^{-.2}$$

and

$$P(2 \text{ or more errors }) = 1 - P(0) - P(1) = 1 - e^{-.2} - .2e^{-.2}.$$

(4.58) Just plugging in numbers. Should see that (a) and (c) reasonable approximations while the situation in (b) is a lousy approximation (for obvious reasons).

(4.60) This is Bayes:

$$\begin{aligned} &P(\text{ works } | 2 \text{ colds }) \\ &= \frac{P(2 \text{ colds } | \text{ works })P(\text{ works})}{P(2 \text{ colds } | \text{ works })P(\text{ works}) + P(2 \text{ colds } | \text{ does not work })P(\text{ does not work})}. \end{aligned}$$

We have

$$P(\text{ 2 colds } | \text{ works }) = P(\text{ a Poisson}(3) = 2) = (9/2)e^{-3}$$

and

$$P(2 \text{ colds } | \text{ does not work }) = P(\text{ a Poisson}(5) = 2) = (25/2)e^{-5}$$

and so the desired probability is

$$\frac{27e^{-3}}{27e^{-3} + 25e^{-5}}.$$

(4.62) Want to approximate this (hopefully) approximate Binomial by a Poisson. The trials correspond to the 20 pairs around the table. The probability of a success is the chance that any such pair is married, or 1/19. Then we would say

$$P(\text{ no pairs married }) \approx P(\text{Poisson}(20/19) = 0) = e^{-20/19}.$$

This is about .349 which is pretty close to the exact answer arrived at by a laborious combinatorial computation.

(Th 4.6) Say N takes values $0, 1, 2, \ldots$. Then

$$\begin{aligned} E[N] &= \sum_{k=1}^{\infty} k P(N = k) \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^{k} P(N = k) \\ &= \sum_{j=1}^{\infty} \sum_{k=j}^{\infty} P(N = k) \\ &= \sum_{j=1}^{\infty} P(N \geq j). \end{aligned}$$

(TH 4.18) Want to solve

$$\frac{d}{d\lambda} P(X = k) = 0$$

for X a Poisson(λ) r.v.. This gives $\lambda^k - k\lambda^{k-1}$ and so $\lambda = 0$ which is senseless or $\lambda = k$ as advertised.

(TH 4.27) The "verbal argument" is: X is the time in takes until a coin sees its first H. Knowing you've seen at least n T's and asking that $X = n + k$ means you see $k - 1$ more T's then an H. The latter is asking a "new" geometric to be equal to k; a coin has no memory.

Alternatively, you can compute:

$$\begin{aligned} P(X = n + k | X \geq n) &= \frac{P(X = n + k)}{P(X \geq n)} \\ &= \frac{p(1-p)^{n+k-1}}{\sum_{j=n}^{\infty} p(1-p)^j} \\ &= \frac{p(1-p)^{n+k-1}}{p(1-p)^n \sum_{j=0}^{\infty} p(1-p)^j} \\ &= \frac{p(1-p)^{n+k-1}}{p(1-p)^n/(1-(1-p))} = p(1-p)^{k-1}. \end{aligned}$$