Introduction to Probability
HW4 Solutions

(3.53) $P(\#1 \text{ works} \mid \text{system works}) =$

$$\frac{P(\#1 \text{ works } \cap \text{ system works })}{P(\text{ system works })} = \frac{P(\#1 \text{ works })}{1 - P(\text{ system broke })} = \frac{1/2}{1-(1/2)^n}.$$

Here line two follows as the system works if any one component works, and line three follows by independence.

(3.58) (a) The procedure is equivalent to conditioing on seeing HT or TH:

$$\begin{aligned} P(\text{say "Heads"} \mid \text{see HT or TH }) &= \frac{P(\text{ see TH })}{P(\text{ see HT or TH })} \\ &= \frac{P(\text{ see TH })}{P(\text{ see HT}) + P(\text{see TH })} = 1/2. \end{aligned}$$

Thus the procedure produces a fair coin flip.

(b) No way this works. Imagine the coin is Tails with probability 0.9999999. Then there is just that probability that the first flip is Tails, which implies that you will say "Heads" with that same probability.

(3.86) Any subset of n elements may be denoted as a string of length n of 0's or 1's: 0 indicating that element is not in the subset, and 1 indicating that it is. Note also each of the 2^n subsets is equally likely: the prob of any string is 2^{-n}.

So we have for (a):

$$\begin{aligned} P(A \subset B) &= \sum_{k=0}^{n} P(A \subset B | N(B) = k) P(N(B) = k) \\ &= \sum_{k=0}^{n} (\frac{1}{2})^{n-k} \cdot \binom{n}{k} (\frac{1}{2})^n \\ &= (\frac{1}{2})^n \sum_{k=0}^{n} \binom{n}{k} (1)^k (\frac{1}{2})^{n-k} \\ &= (\frac{1}{2})^n (1 + \frac{1}{2})^n = (3/4)^n \end{aligned}$$

by the Binomial Theorem in the last line. Here we have used

$$P(N(B) = k) = \binom{n}{k} (\frac{1}{2})^n$$

as there are $\binom{n}{k}$ choices of 1's and each resulting string occurs with prob $(1/2)^n$. We also have

$$P(A \subset B | N(B) = k) = (\frac{1}{2})^{n-k}$$

because the string representing A must have a zero in each place that string representing B does and there are $n-k$ such places.

(b) Since $P(A \cap B = \phi) = P(A \subset B^c)$ this is the same question as part (a).

(TH 3.7 (b)) First note that

$$\begin{aligned} P(\text{R goes first}) &= P(R,G,B) + P(R,B,G) \\ &= P(R,G,B|\text{ last is B})P(\text{ last is B}) + P(R,B,G|\text{ last is G})P(\text{ last is G}) \end{aligned}$$

where we read $\{R, G, B\}$ as "first R goes, then G, then B". Now a little thought will convince you that

$$P(R,G,B|\text{ last is B}) = P(R,G)$$

which is a question about just two colors which we did in (TH 3.7 (a)). Putting it together we have

$$\begin{aligned} P(\text{R goes first}) &= P(R,G)P(\text{ last is B}) + P(R,B)P(\text{ last is G}) \\ &= \frac{g}{r+g} \cdot \frac{b}{r+b+g} + \frac{b}{r+b} \cdot \frac{g}{r+b+g}. \end{aligned}$$

(4.12) X takes values $\{2, 3, 4, 0, -2, -3, -4\}$.

For (a), use independence:

$$P(2) = P(show1)P(see1)P(say1)P(hear2) = 1/16.$$

Now, $P(4)$ is the same, and for $P(3)$ you change the rolls of 1 and 2 to find $P(3) = 1/8$. Clearly $P(k) = P(-k)$ and so $P(0) = 1/2$.

(b) No one can win: $P(0) = 1$.

(4.20) (a) $P(X > 0) = P(win) + P(lose, win, win) = (18/38) + (20/38)(18/38)^2$.

(b) $E[X] = 1 \cdot P(X = 1) - 1 \cdot P(X = -1) - 3 \cdot P(X = -3)$
$= P(X > 0) - (P(lose, win, lose) + P(lose, lose, win)) - 3P(lose, lose, lose)$
$= (18/38) + (20/38)(18/38)^2 - 2(18/38)(20/38)^2 - 3(20/38)^3$.

(4.22) X takes values 2 or 3.

$$P(X = 2) = P(A, A) + P(B, B) = p^2 + (1 - p)^2$$

and

$$P(X = 3) = P(A, B, A) + P(B, A, A) + P(B, A, B) + P(A, B, B) = 2p^2(1 - p) + 2p(1 - p)^2.$$

So $E[X] = 2(p^2 + (1 - p)^2) + 6(p^2(1 - p) + p(1 - p)^2)$ which is maximized at $p = 1/2$.