Introduction to Probability
HW3 Solutions

(3.1) Directly from the definition of conditional probability:

$$\begin{aligned} P(\text{at least one six} \mid \text{each roll different}) &= \frac{P(\text{one six and the other not})}{P(\text{each roll different})} \\ &= \frac{10/36}{30/36} = 1/3. \end{aligned}$$

(3.5) Condition to find

$$\begin{aligned} P(\circ\circ\bullet\bullet) &= P(\circ\circ\bullet\bullet \mid \text{pick 2 } \circ'\text{s and 2 } \bullet'\text{s}) P(\text{ pick 2 } \circ'\text{s and 2 } \bullet'\text{s}) \\ &= \frac{1}{\binom{4}{2}} \cdot \frac{\binom{6}{2}\binom{9}{2}}{\binom{15}{4}} = 6/91. \end{aligned}$$

(3.24) Point is to understand what the conditional info is telling you. To start there are 4 scenarios: GB, BG, GG, BB. Noting that the first two are different is important.

a) Knowing that gold paint was used rules out BB. Therefore, the desired event is now one of three possibilities and the conditional probability is 1/3.

b) This is a question about just one ball, and for either we have that $P(G) = 1/2$.

(3.28) Using that the conditional space is all lists of 52 cards where we see no ace until card (or place) number twenty we compute:

a) $P($ card twenty-one is Ace of Spades $\mid$ first ace on card 20)

$$= \frac{48 \cdot 47 \cdots 30 \cdot 3 \cdot 1 \cdot 31 \cdot 30 \cdots 1}{48 \cdot 47 \cdots 30 \cdot 4 \cdot 32 \cdot 31 \cdot 30 \cdots 1} = \frac{3}{4 \cdot 32}.$$

b) $P($card twenty-one is Two of Clubs $\mid$ first ace on card 20)

$$= \frac{47 \cdot 46 \cdots 29 \cdot 4 \cdot 1 \cdot 31 \cdot 30 \cdots 1}{48 \cdot 47 \cdots 30 \cdot 4 \cdot 32 \cdot 31 \cdot 30 \cdots 1} = \frac{29}{48 \cdot 32}$$

as now both aces and the two of clubs (that is, five cards) must be avoided in the first nineteen places comprising the numerator.

(3.38) This is "straight-up" Bayes:

$$\begin{aligned} P(\text{ coin is T} \mid \text{see White}) &= P(\text{ used urn B} | W) \\ &= \frac{P(W|\text{ urn B})P(\text{ urn B })}{P(W|\text{ urn B})P(\text{ urn B }) + P(W|\text{ urn A})P(\text{ urn A })} \\ &= \frac{3/15}{5/12 + 3/15} = 12/37. \end{aligned}$$

(3.57) A question in independence:

a) $P($ no move after 2 days$) = P(UD) + P(DU) = 2P(U) \cdot P(D) = 2p(1-p)$.

b) $P($ up one after 3 days$) = P(UUD) + P(UDU) + P(DUU) = 3p^2(1-p)$

c) $P($ day one is up $\mid$ up one after 3 days$) =$

$$\frac{P(UDU) + P(UUD)}{P(UDU) + P(UUD) + P(DUU)} = 2/3.$$

(TH 3.6) The real point is to realize that if $\{E_1, E_2, \ldots, E_n\}$ are all independent then so are $\{E_1^c, E_2^c, \ldots, E_n^c\}$ and therefore

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} E_i\right) &= 1 - P\left(\bigcap_{i=1}^{n} E_i^c\right) \\ &= 1 - \prod_{i=1}^{n} P(E_i^c) = 1 - \prod_{i=1}^{n}\Big(1 - P(E_i)\Big) \end{aligned}$$

where DeMorgan is used in line one.

(TH 3.7 (a)) The hint is to think about lying all the balls in a row (after removing them all, where you line them up in the order that they were removed). Then the question becomes: what is the probability that the last several (however many) are white. This turns out to be the same as asking what is the probability the last is white:

$$\begin{aligned} & P(\text{ last few are white}) \\ &= P(\text{ last few are white } \cap \text{ last is white}) + P(\text{ last few are white } \cap \text{ last is black}) \\ &= P(\text{ last few are white } \mid \text{ last is white}) \cdot P(\text{ last is white }) + 0 \\ &= 1 \cdot P(\text{ last is white }) + 0 \\ &= \frac{n}{n+m}. \end{aligned}$$