MATH 2400: CALCULUS 3

MAY 9, 2007

FINAL EXAM

I have neither given nor received aid on this exam.

Name:____

◯ **001** E. Kim .... (9am)

◯ **002** E. Angel .... (10am)

◯ **003** I. Mishev .... (11am)

◯ **004** M. Daniel .... (12am)

◯ **005** A. Gorokhovsky ...... (1pm)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

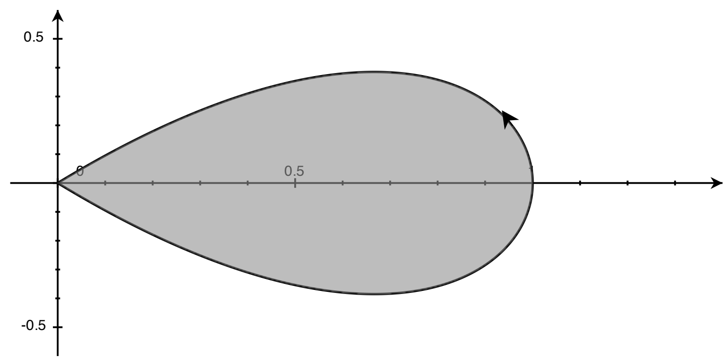
DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	15 pts	
2	15 pts	
3	15 pts	
4	15 pts	
5	20 pts	
6	15 pts	
7	30 pts	
8	15 pts	
9	15 pts	
10	30 pts	
11	15 pts	
TOTAL	200 pts	

1. (15 pt) Find the area of the region enclosed by the curve

$$x = 1 - t^2,\ y = t(1-t^2),\ -1 \le t \le 1$$

(see the picture below)



Clearly the region is symmetric about the x-axis, so we have $A = 2\int_0^1 f(x)\,dx = 2\int_0^1 y\,dx$. Now $y = t(1-t^2)$, $dx = -2t\,dt$, and the top half of the curve is traced from time $t = 0$ to $t = 1$. Thus, $A = 2\int_0^1 t(1-t^2)(-2t)\,dt = -4\int_0^1 (t^2 - t^4)\,dt = \frac{1}{12}$.

2. (15 pt) Find the equation for the plane tangent to the paraboloid $z = 2x^2 + 3y^2$ that is also parallel to the plane $4x - 3y - z = 10$

Write $G(x,y,z) = 2x^2 + 3y^2 - z$, so $\nabla G(x,y,z) = \langle 4x, 6y, -1\rangle$. This tells us that $\langle 4x, 6y, -1\rangle$ is a normal vector for the tangent plane to our surface at the point (x,y,z). For the tangent plane at a point to be parallel to $4x - 3y - z = 10$, we need $\langle 4x, 6y, -1\rangle$ to be parallel to $\langle 4, -3, -1\rangle$. This will certainly happen when $x = 1$ and $y = -\frac{1}{2}$. Solving for z in the equation for our surface, we get that the tangent plane at $(1, -\frac{1}{2}, \frac{11}{4})$ is parallel to $4x - 3y - z = 10$. At this point the equation for the tangent plane will be given by $4(x-1) - 3(y+\frac{1}{2}) - (z - \frac{11}{4}) = 0$.

3. (15 pt) Find the flux of $\mathbf{F}(x,y,z) = (x+y)\mathbf{i} + (y+z)\mathbf{j} + (z+x)\mathbf{k}$ across the portion of the plane $x+y+z=1$ in the first octant oriented by unit normals with positive components.

We know $\Phi = \iint\limits_{\sigma} \mathbf{F} \bullet \mathbf{n}\, dS$. Let $u = x$ and $v = y$. Then our surface is given by

$$\mathbf{r}(u,v) = \langle u, v, 1-u-v \rangle\,, \quad 0 \le u \le 1\,,\, 0 \le v \le 1\,.$$

Next,

$$\mathbf{n}\, dS = (\mathbf{r}_u \times \mathbf{r}_v)\, dA = (\langle 1, 0, -1\rangle \times \langle 0, 1, -1\rangle)\, dA = \langle 1,1,1\rangle\, dA\,.$$

Then

$$\begin{aligned}
\Phi &= \iint\limits_{\sigma} \mathbf{F} \bullet \mathbf{n}\, dS \\
&= \int_0^1 \int_0^1 \langle u+v, v+(1-u-v), (1-u-v)+u\rangle \bullet \langle 1,1,1\rangle\, dv\, du \\
&= \int_0^1 \int_0^1 2\, dv\, du \\
&= 2
\end{aligned}$$

4. (15 pt) Evaluate the integral

$$\int_0^1 \int_{\sqrt{x}}^1 \cos\left(y^3\right) dy\, dx$$

by changing the order of integration.

$$\begin{aligned}
\int_0^1 \int_{\sqrt{x}}^1 \cos\left(y^3\right) dy\, dx &= \int_0^1 \int_0^{y^2} \cos\left(y^3\right) dx\, dy \\
&= \int_0^1 \cos\left(y^3\right) y^2\, dy \\
&= \frac{1}{3} \sin(y^3)\Big|_0^1 \\
&= \frac{\sin(1)}{3}
\end{aligned}$$

5. A wire 12 cm long is cut into three or fewer pieces, with each piece bent into a square.

(a) (10 pt) What is the minimal total area of the squares? Justify your answer.

Let x, y, z denote the lengths of the three segments of the wire after it is cut, so $x+y+z = 12$. Bending each segment into a square yields a total area of $A = \left(\frac{x}{4}\right)^2 + \left(\frac{y}{4}\right)^2 + \left(\frac{z}{4}\right)^2$. Replacing z with $12 - x - y$ (and simplifying), we have

$$A(x, y) = \frac{1}{8}(x^2 + y^2 + xy - 12x - 12y + 72)\,.$$

Now, $x+y+z = 12$ forces x and y to be confined to the region in the xy-plane, R, which is the triangle with vertices $(0, 0), (12, 0), (0, 12)$ (this is just the projection of surface $x + y + z = 12$, with $x, y, z \geq 0$ onto the xy-plane). We now minimize and maximize A over R.

Find Interior Critical Points: We simultaneously solve $A_x = \frac{1}{8}(2x + y - 12) = 0$ and $A_y = \frac{1}{8}(2y + x - 12) = 0$ to find the one interior critical point $(4, 4, 4)$.

Test Critical Points and Boundary Points:
On the portion of the boundary of R that lies on the x-axis, we have $y = 0$ and $0 \leq x \leq 12$. Substituting into A we get $A = \frac{1}{8}(x^2 - 12x + 72)$, $0 \leq x \leq 12$. The absolute extrema of $\frac{1}{8}(x^2 - 12x + 72)$ on $0 \leq x \leq 12$ occur when $x = 0, 12$, so we must consider the points $(0, 0, 12), (12, 0, 0)$.

On the portion of the boundary of R that lies on the y-axis, we have $x = 0$ and $0 \leq y \leq 12$. Similar to the previous case, we find that we must consider the points $(0, 0, 12), (0, 12, 0)$.

Finally, on the portion of the boundary of R that lies on the line $y = 12 - x$, we have $y = 12 - x$ and $0 \leq x \leq 12$. Substituting into A we get $A = \frac{1}{8}(x^2 - 12x + 72)$, $0 \leq x \leq 12$. As before, the absolute extrema of $\frac{1}{8}(x^2 - 12x + 72)$ on $0 \leq x \leq 12$ occur when $x = 0, 12$, so we must consider the points $(0, 12, 0), (12, 0, 0)$.

Testing all the points that we have listed, we find that the minimum total area is $3\,\text{cm}^2$ occurring when each segment is of length 4 cm.

(b) (10 pt) What is the maximal total area of the squares? Justify your answer.

Using the above information, we find that the maximum total area is $9\,\text{cm}^2$ occurring when we make no cut and have only one segment of length 12 cm.

6. (15 pt) Evaluate $\oint_C \mathbf{F} \bullet d\mathbf{r}$, where $\mathbf{F}(x,y,z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$ and C is the triangle in the plane $z = \frac{1}{2}y$ with vertices $(2,0,0)$, $(0,2,1)$ and $(0,0,0)$ with a counterclockwise orinetation looking down the positive z-axis.

We use Stoke's Theorem, so $\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_\sigma (\text{curl}\mathbf{F}) \bullet \mathbf{n} dS$. Let $u = x$ and $v = y$. Then our surface, σ, is given by

$$\mathbf{r}(u,v) = \langle u, v, \frac{1}{2}v \rangle, \quad 0 \le u \le 2, 0 \le v \le 2-u.$$

Next,

$$\mathbf{n}\, dS = (\mathbf{r}_u \times \mathbf{r}_v)\, dA = (\langle 1,0,0\rangle \times \langle 0,1,\frac{1}{2}\rangle)\, dA = \langle 0, -\frac{1}{2}, 1\rangle\, dA.$$

Note that we have to choose between $\pm\langle 0, -\frac{1}{2}, 1\rangle$ to correctly orient the surface, but the '+' is chosen to correspond to the orientation of the curve. Then

$$\begin{aligned}
\oint_C \mathbf{F} \bullet d\mathbf{r} &= \iint_\sigma (\text{curl}\mathbf{F}) \bullet \mathbf{n} dS \\
&= \int_0^2 \int_0^{2-u} \langle -4, -6, 6v\rangle \bullet \langle 0, -\frac{1}{2}, 1\rangle\, dv\, du \\
&= \int_0^2 \int_0^{2-u} (3+6v)\, dv\, du \\
&= \int_0^2 (-3u^2 + 9u - 6)\, du \\
&= -11
\end{aligned}$$

7. The helix $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $P(1,0,0)$.

(a) (15 pt) Find the tangent line to each curve at P.

Considering when $\mathbf{r}_2$ is at P, we see that the intersection occurs when $t = 0$. Now, $\mathbf{r}_1' = \langle -\sin t, \cos t, 1\rangle$, so the tangent line to the curve given by $\mathbf{r}_1$ at $t = 0$ will have direction given by $\langle 0,1,1\rangle$. Thus the tangent line is given by $\langle 1,0,0\rangle + t\langle 0,1,1\rangle = \langle 1,t,t\rangle$. Similarly, we find that the tangent line to the curve given by $\mathbf{r}_1$ at $t = 0$ is given by $\langle 1+t, 0, 0\rangle$.

(b) (15 pt) Find the angle between the tangent lines at P.

The angle between them, θ, can be found by solving

$$\cos\theta = \frac{\mathbf{d}_1 \bullet \mathbf{d}_2}{||\mathbf{d}_1||\,||\mathbf{d}_2||}$$

where $\mathbf{d}_1$ and $\mathbf{d}_2$ are direction vectors for the lines. Thus

$$\cos\theta = \frac{\langle 0,1,1\rangle \bullet \langle 1,0,0\rangle}{||\langle 0,1,1\rangle||\,||\langle 1,0,0\rangle||} = 0$$

so $\theta = \frac{\pi}{2}$.

8. (15 pt) Evaluate $\iint\limits_R \sin\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)dA$, where R is the triangular region whose vertices are $(0,0)$, $(2,0)$ and $(1,1)$.

We start with a change of coordinates. Set $u = \frac{1}{2}(x+y)$ and $v = \frac{1}{2}(x-y)$, so that $x = u+v$ and $y = u-v$. The region R transforms to the uv-region S which is a triangle with vertices $(0,0)$, $(1,0)$ and $(1,1)$. Calculating the Jacobian, we find that $\frac{\partial(x,y)}{\partial(u,v)} = -2$. Thus,

$$\begin{aligned}\iint\limits_R \sin\left(\frac{1}{2}(x+y)\right)\cos\left(\frac{1}{2}(x-y)\right)\,dA &= \iint\limits_S \sin u\cos v\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dA \\ &= \int_0^1\int_0^u 2\sin u\cos v\,dv\,du \\ &= \int_0^1 2\sin^2 u\,du \\ &= \int_0^1 (1-\cos(2u))\,du \\ &= -\frac{1}{2}\sin 2 + 1\end{aligned}$$

9. (15 pt) Consider $\int_{(0,0)}^{(3,2)}(2xe^y+1)dx + x^2e^y dy$. Show that the integral is independent of path and evaluate the integral using the Fundamental Theorem of Line Integrals.

To check that the vector field appearing in this integral is conservative it is enough to check the equality of the partial derivatives, since it is defined in the whole plane:

$$\frac{\partial}{\partial x}(x^2e^y) = 2xe^y = \frac{\partial}{\partial y}(2xe^y+1)$$

Let ϕ be the potential. Then $\frac{\partial\phi}{\partial x} = 2xe^y+1$, $\phi = \int(2xe^y+1)dx = x^2e^y + x + C(y)$. $\frac{\partial\phi}{\partial y} = x^2e^y$. Hence $C'(y) = 0$, C is a constant, and we can choose $C = 0$. Then $\phi = x^2e^y + x$ and

$$\int_{(0,0)}^{(3,2)}(2xe^y+1)dx + x^2e^y dy = \phi(3,2) - \phi(0,0) = 9e^2 + 3$$

10. Find

(a) (15 pt) the equation of the plane through $P(1,-1,2)$, $Q(2,1,3)$, and $R(-1,2,-1)$.

We have: $\overrightarrow{PQ} = \langle 1,2,1\rangle$, $\overrightarrow{PR} = \langle -2,3,-3\rangle$,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ -2 & 3 & -3 \end{vmatrix} = -9\mathbf{i} + \mathbf{j} + 7\mathbf{k}$$

Therefore the equation of the plane is $-9(x-1) + (y+1) + 7(z-2)$ or

$$-9x + y + 7z - 4 = 0$$

(b) (15 pt) parametric equations of the line of intersection of the planes

$$x + 2y + z = 1 \text{ and } x - y + 2z = -8.$$

Normal vectors of the planes are $\langle 1,2,1\rangle$ and $\langle 1,-1,2\rangle$. The direction vector of the intersection line is given by the cross-product of the normal vectors, i.e. by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$$

Now look for a point in the intersection, e.g. the one which has $z = 0$. We then have $x+2y = 1$, $x - y = -8$. From this $x = -5$, $y = 3$, i.e. the point $(-5,3,0)$ is in the intersection of the planes. From this, the parametric equations of the line are

$$\begin{aligned} x &= -5 + 5t \\ y &= 3 - t \\ z &= -3t \end{aligned}$$

11. (a) (1 pt) Write a formula expressing the directional derivative $D_{\mathbf{n}}f$ of a function f along a unit vector $\mathbf{n}$ in terms of ∇f and $\mathbf{n}$.

$$D_{\mathbf{n}}f = \nabla f \bullet \mathbf{n}$$

(b) (14 pt) Let σ be the sphere $x^2 + y^2 + z^2 = 1$, let $\mathbf{n}$ be an outward unit normal, and let $D_{\mathbf{n}}f$ be the directional derivative of $f(x, y, z) = x^2 + e^x \cos y + y^2 + z^2$. Evaluate $\iint\limits_{\sigma} D_{\mathbf{n}}f dS$.

$$\iint\limits_{\sigma} D_{\mathbf{n}}f dS = \iint\limits_{\sigma} \nabla f \bullet \mathbf{n} dS = \iiint\limits_{G} \mathbf{div} \nabla f dV$$

where G is the interior of the sphere $x^2 + y^2 + z^2 = 1$.

Now $\nabla f = \langle 2x + e^x \cos y, -e^x \sin y + 2y, 2z \rangle$, and $\mathbf{div} \nabla f = 6$. Therefore

$$\iint\limits_{\sigma} D_{\mathbf{n}}f dS \iiint\limits_{G} \mathbf{div} \nabla f dV = 6 Volume(G) = 6 \left(\frac{4\pi}{3} \right) = 8\pi$$