

MATH 2400: CALCULUS 3

April 18, 2007

MIDTERM 3

I have neither given nor received aid on this exam.

Name: _____

001 E. KIM (9AM)

004 M. DANIEL (12AM)

002 E. ANGEL (10AM)

005 A. GOROKHOVSKY (1PM)

003 I. MISHEV (11AM)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	14 pts	
2	12 pts	
3	14 pts	
4	12 pts	
5	12 pts	
6	12 pts	
7	24 pts	
TOTAL	100 pts	

1. (14 pt) Compute $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$

Convert to polar:

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 r dr d\theta = \frac{\pi}{8}$$

2. (12 pt) Find the volume of the solid that lies inside the sphere $\rho = 3$ and above the cone $r = z$ ($\phi = \frac{\pi}{4}$).

Use spherical:

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^3 \rho^2 \sin \phi d\rho d\phi d\theta = 9 \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \sin \phi d\phi d\theta = 18\pi - 9\pi\sqrt{2}$$

3. (14 pt) Find the volume of the solid bounded by

$$z = 6, \quad z = 2y, \quad y = x^2, \quad y = 2 - x^2.$$

$$\begin{aligned} V &= \int_{-1}^1 \int_{x^2}^{2-x^2} \int_6^{2y} dz \, dy \, dx \\ &= \int_{-1}^1 [6(2-x^2) - (2-x^2)^2] - [6x^2 - x^4] \, dx \\ &= \int_{-1}^1 8 - 8x^2 \, dx \\ &= \frac{32}{3} \end{aligned}$$

4. (12 pt) Find the surface area of the portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

We are concerned with the portion of the surface ABOVE the xy -plane, so the cone can be represented by $z = \sqrt{4x^2 + 4y^2}$. Then,

$$\begin{aligned} SA &= \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA \\ &= \iint_R \sqrt{\left(\frac{4x}{\sqrt{4x^2 + 4y^2}}\right)^2 + \left(\frac{4y}{\sqrt{4x^2 + 4y^2}}\right)^2 + 1} \, dA \\ &= \iint_R \sqrt{\left(\frac{16x^2}{4x^2 + 4y^2}\right) + \left(\frac{16y^2}{4x^2 + 4y^2}\right) + 1} \, dA \\ &= \iint_R \sqrt{5} \, dA \\ &= \int_0^1 \int_{x^2}^x \sqrt{5} \, dy \, dx \\ &= \frac{\sqrt{5}}{6} \end{aligned}$$

5. (12 pt) Find the area of the region bounded by the curves

$$y = x^2, \quad y = 3x^2, \quad xy = 1, \quad xy = 4.$$

Do a change of variables. Set $u = \frac{y}{x^2}$ and $v = xy$. The transformed region is now bounded by the curves

$$u = 1, \quad u = 3, \quad v = 1, \quad v = 4.$$

Solving for x and y , we find

$$x = u^{-\frac{1}{3}}v^{\frac{1}{3}}, \quad y = u^{\frac{1}{3}}v^{\frac{2}{3}}.$$

Calculating the Jacobian of the transformation, we find that

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{3}u^{-1}.$$

Now,

$$A = \int_1^4 \int_1^3 \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv = \int_1^4 \int_1^3 \frac{1}{3}u^{-1} du dv = \ln(3).$$

6. (12 pt) Evaluate $\int_C yz^2 ds$ along the helix

$$C : \mathbf{r}(t) = 2t \mathbf{i} + 3 \sin t \mathbf{j} + 3 \cos t \mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}.$$

$$\begin{aligned} \int_C yz^2 ds &= \int_0^{\frac{\pi}{2}} (3 \sin t)(3 \cos t)^2 \sqrt{4 + 9 \cos^2 t + 9 \sin^2 t} dt \\ &= 27\sqrt{13} \int_0^{\frac{\pi}{2}} \sin t \cos^2 t dt \quad (\text{now use } u = \cos t) \\ &= 27\sqrt{13} \left[-\frac{1}{3} \cos^3 t \right]_0^{\frac{\pi}{2}} \\ &= 9\sqrt{13} \end{aligned}$$

7. Given the force field $\mathbf{F}(x, y) = e^{2y} \mathbf{i} + (1 + 2xe^{2y}) \mathbf{j}$ and curve

$$C : \mathbf{r}(t) = te^t \mathbf{i} + (1 + t) \mathbf{j}, \quad 0 \leq t \leq 1,$$

(a) (12 pt) Find the potential function for \mathbf{F} .

Set $f = e^{2y}$ and $g = 1 + 2xe^{2y}$. Since $\frac{\partial f}{\partial y} = 2e^{2y} = \frac{\partial g}{\partial x}$, \mathbf{F} is conservative. We want to find a function ϕ such that $\nabla\phi = \mathbf{F}$. Now,

$$\phi = \int f dx = xe^{2y} + k(y).$$

Taking the derivative with respect to y , we find

$$\frac{\partial\phi}{\partial y} = 2xe^{2y} + k'(y).$$

This must be equal to g , so $k'(y) = 1$. Now integrate $k'(y)$ with respect to y to find that $k(y) = y + K$, so

$$\phi = xe^{2y} + y + K.$$

Note that any choice for K will make ϕ a potential function for \mathbf{F} . In particular, we can choose $K = 0$, so

$$\phi = xe^{2y} + y.$$

(b) (12 pt) Find the work done moving a particle along C in the force field \mathbf{F} .

Note that the curve starts at $(0, 1)$ and ends at $(e, 2)$, i.e. the points corresponding to $\mathbf{r}(0)$ and $\mathbf{r}(1)$. Using the Fundamental theorem of line integrals, we find that

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \phi(e, 2) - \phi(0, 1) = e^5 + 1.$$