MATH 4510: Introduction to Probability

November 19, 2008

Midterm 2

I have neither given nor received aid on this exam.

Name:____

In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	10 pts	
2	10 pts	
3	15 pts	
4	15 pts	
Total	50 pts	

1. (10pts) The average number of Atlantic hurricanes which occur in a given year is 8.4. Using the Poisson paradigm, find the (approximate) probability that there will be more than 20 hurricanes in the next 3 years.

SOLUTION:

If $N(t)$ represents the number of Atlantic hurricanes which occur in the interval $[0, t]$ (where t is measured in years), and if we approximate $N(t)$ by a Poisson random variable with parameter λ, then

$$P\{N(t) = k\} = e^{-\lambda t}\frac{(\lambda t)^k}{k!}.$$

Here $\lambda = 8.4$ is the mean number of hurricanes in a year and $t = 3$.

Thus,

$$P\{N(3) > 20\} = \sum_{k=21}^{\infty} e^{-8.4\times 3}\frac{(8.4\times 3)^k}{k!}$$

2. (10pts) A news boy purchases papers at 10 cents and sells them at 15 cents. If he is not allowed to return unsold papers, and his daily demand is approximated by a uniform random variable on $[50, 80]$, approximately how many papers should he purchase to maximize his expected daily profit?

SOLUTION:

Let y denote the number of papers he orders. Clearly, if he is to maximize his profit, y must be between 50 and 80. If X is his daily demand, then his daily profit, Q is given by

$$Q(X) = \begin{cases} 15X - 10y & \text{if } 50 \le X \le y \le 80 \\ 5y & \text{if } 50 \le y < X \le 80 \end{cases}$$

It follows that

$$\begin{aligned} E[Q] &= \int_{50}^{80} \frac{Q(x)}{30} dx \\ &= \int_{50}^{y} \frac{5y}{30}\, dx + \int_{y}^{80} \frac{1}{30}(15x - 10y)\, dx \\ &= \frac{y(y-50)}{6} + \frac{80^2 - y^2}{4} - \frac{y(80-y)}{3} \\ &= \frac{1}{4}(y^2 - 140y + 6400) \end{aligned}$$

To maximize his daily profit, we differentiate this with respect to y and solve for 0:

$$\frac{d}{dy} E[Q] = \frac{1}{4}(2y - 140) \qquad \text{which equals 0 when } y = 70.$$

Thus, he should buy 70 papers to maximize his expected daily profit.

3. (15pts) For some constant c, the random variable X has probability density function

$$f(x) = \begin{cases} \frac{c}{x^3} & \text{if } x > 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c.

SOLUTION:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_1^{\infty} x^{-3}\,dx = c\left[\frac{x^{-2}}{-2}\right]_1^{\infty} = \frac{c}{2}.$$

Thus, $c = 2$.

(b) Find the distribution function of X.

SOLUTION:

If $x < 1$ we have $F(x) = 0$. Otherwise,

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_1^x 2t^{-3}\,dt = \left[-t^{-2}\right]_1^x = 1 - x^{-2}.$$

That is,

$$F(x) = \begin{cases} 0 & \text{if} x < 1; \\ 1 - \frac{1}{x^2} & \text{if } x \geq 1. \end{cases}$$

(c) Find the density function of e^X.

SOLUTION:

Let $Y = e^X$. Then Y is random variable that takes on values between e and ∞. Let F_Y be the distribution function of Y. Then

$$F_Y(x) = P\{Y \leq x\} = P\{e^X \leq x\} = \begin{cases} 0 & \text{if } x < e; \\ P\{X \leq \log x\} & \text{if } x \geq e. \end{cases}$$

From (b), if $x \geq e$,

$$F_Y(x) = P\{X \leq \log x\} = 1 - \frac{1}{(\log x)^2}.$$

It follows that, if $x \geq e$, the density function of Y, $f_Y(x)$ is given by

$$f_Y(x) = \frac{d}{dx} F_Y(x) = \frac{2}{x(\log x)^3}.$$

That is,

$$f_Y(x) = \begin{cases} 0 & \text{if } x < e; \\ \dfrac{2}{x(\log x)^3} & \text{if } x \geq e. \end{cases}$$

4. (15pts) Suppose N is a standard normal random variable with density function f.

(a) Prove that $\int_{-\infty}^{\infty} f(x) = 1$.

SOLUTION:

Let

$$I = \int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx.$$

Then

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy$$

Switching to polar coordinates, $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$ so that

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2/2} r\,dr = \int_0^{\infty} e^{-r^2/2} r\,dr.$$

Using u-substitution on the latter integral, with $u = r^2/2$ and $r\,dr = du$, we find

$$I^2 = \int_0^{\infty} e^{-u}\,du = \left[-e^{-u}\right]_0^{\infty} = 1.$$

Thus $I^2 = 1$, and since I is clearly positive, we have $I = 1$.

(b) Prove that $E[N] = 0$.

SOLUTION:

$$E[N] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x\,dx$$

This integral must equal 0 since the integrand is an odd function. In other words,

$$\begin{aligned} E[N] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-x^2/2} x\,dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2/2} x\,dx \\ &= -\frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2/2} x\,dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2/2} x\,dx = 0. \end{aligned}$$

(c) Prove that $\text{Var}(N) = 1$.

SOLUTION:

$$\text{Var}(N) = E[N^2] - E[N]^2 = E[N^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^2 \, dx$$

We use integration by parts with $u = x$ and $dv = xe^{-x^2/2}$ so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} x^2 \, dx = \left[-\frac{1}{\sqrt{2\pi}} x e^{-x^2/2} \right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx$$

The first term is 0, and the second is 1 by part (a).