MATH 2400: CALCULUS 3

March 14, 2007

MIDTERM 2

I have neither given nor received aid on this exam.

Name:____

- ◯ **001** E. Kim .... (9AM)
- ◯ **002** E. Angel .... (10AM)
- ◯ **003** I. Mishev .... (11AM)
- ◯ **004** J. Boisvert .... (12AM)
- ◯ **005** A. Gorokhovsky ...... (1PM)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	18 pts	
2	16 pts	
3	10 pts	
4	16 pts	
5	10 pts	
6	10 pts	
7	10 pts	
8	10 pts	
TOTAL	160 pts	

1. Consider $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$.

(a) (6 pts) Compute the limit along the line $y = x$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along } y=0}} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{x^4} = 0$$

(b) (6 pts) Compute the limit along the parabola $y = x^2$.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{Along } y=x^2}} \frac{2x^2y}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^4}{x^4+x^4} = \lim_{(x,y)\to(0,0)} 1 = 1$$

(c) (6 pts) Based on parts (a) and (b), what is $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^4+y^2}$?

The limit does not exist because the limits along different paths to $(0,0)$ are different.

2. Given a function $f(x, y) = y\cos(x)$ and a point $P(0, 0)$,

(a) (8 pts) Find the local linear approximation L to the function $f(x, y)$ at the point P.

We have $f_x(P) = -(0)\sin(0) = 0$, and $f_y(P) = \cos(0) = 1$. Therefore,
$L(x, y) = f(P) + f_x(P)(x - x_0) + f_y(P)(y - y_0) = 1(y - 0)$, so $L(x, y) = y$.

(b) (8 pts) Given an additional point $Q(0.004, 0.003)$, use part (a) to approximate $f(Q)$.

$f(Q) \approx L(Q) = 0.003.$

3. (10 pts) Give a **unit** vector that points in the direction of maximum increase for the function $f(x, y, z) = x^3z^3 + y^3z + z - 1$ at $P(1, 1, -1)$.

Our answer will be $\dfrac{\nabla f(P)}{||\nabla f(P)||}$. $f_x = 3x^2z^3$, $f_y = 3y^2z$, $f_z = 3x^3z^2 + y^3 + 1$, so $\nabla f(P) = \langle 3(1)(-1), 3(1)(-1), 3(1)(1)+1+1\rangle = \langle -3, -3, 5\rangle$. Therefore, our answer is $\mathbf{u} = \dfrac{\nabla f(P)}{||\nabla f(P)||} = \dfrac{\langle -3, -3, 5\rangle}{||\langle -3, -3, 5\rangle||} = \dfrac{\langle -3, -3, 5\rangle}{\sqrt{43}}$.

4. (a) (8pts) Find an equation of the tangent plane to the surface S given by $x^2+y^2+z+z^3=4$ at the point $P(1,1,1)$.

$\nabla f(P) = \langle 2(1), 2(1), 1+3(1)^2\rangle = \langle 2,2,4\rangle$ will serve as a normal vector for the plane. This gives an equation for the plane that is $2(x-1)+2(y-1)+4(z-1)=0$, which simplifies to $x+y+2z=4$.

(b) (8pts) Suppose that the point $Q(0.96, 1.02, z)$ is on the surface S from part (a). Give the best approximation to the value of z you can.

The tangent plane approximates the surface close to P. Solving for z in the equation for the tangent plane, we get $z = 2 - \frac{x}{2} - \frac{y}{2}$, so the z value of Q is approximately $2 - \frac{0.96}{2} - \frac{1.02}{2} = 2 - 0.48 - 0.51 = 1.01$.

5. (10 pts) Find and classify all the critical points of the function f defined by

$$f(x,y) = x^3 - y^4 - 3x + 4y + 5$$

We have $f_x = 3x^2 - 3$ and $f_y = -4y^3 + 4$, so the critical points will be the points (x, y) that solve the system $3x^2 - 3 = 0$ and $-4y^3 + 4 = 0$. This means we must have $x = \pm 1$ and $y = 1$, so our critical points are $(\pm 1, 1)$. We now use the Second partials test to classify the points. Now, $f_{xx} = 6x$, $f_{yy} = -12y^2$, and $f_{xy} = 0$, so $D = -72xy^2$. $D(1, 1) < 0$ so $(1, 1)$ is a saddle point. $D(-1, 1) > 0$ and $f_{xx}(-1, 1) < 0$, so $(-1, 1)$ is a relative maximum.

6. (10 pts) Let $\ln(z + y - z^3) = x$. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.

Now,

$$\begin{aligned}
\frac{\partial}{\partial x}\ln(z + y - z^3) &= \frac{\partial}{\partial x}x \\
\frac{1}{z + y - z^3}\frac{\partial}{\partial x}(z + y - z^3) &= 1 \\
\frac{1}{z + y - z^3}\left(\frac{\partial z}{\partial x} + 0 - 3z^2\frac{\partial z}{\partial x}\right) &= 1
\end{aligned}$$

so, $\dfrac{\partial z}{\partial x} = \dfrac{z + y - z^3}{1 - 3z^2} = \dfrac{e^x}{1 - 3z^2}$. Similarly,

$$\begin{aligned}
\frac{\partial}{\partial y}\ln(z + y - z^3) &= \frac{\partial}{\partial y}x \\
\frac{1}{z + y - z^3}\frac{\partial}{\partial y}(z + y - z^3) &= 0 \\
\frac{1}{z + y - z^3}\left(\frac{\partial z}{\partial y} + 1 - 3z^2\frac{\partial z}{\partial y}\right) &= 0
\end{aligned}$$

so, $\dfrac{\partial z}{\partial y} = -\dfrac{1}{1 - 3z^2}$.

7. (10 pts) The sum of three nonnegative numbers x, y and z is 6. How large can their product be?

We want to maximize $P = xyz$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x+y+z = 6$. This means we want to minimize $P = xy(6-x-y)$ over the region R given by $x \geq 0$, $y \geq 0$, $x + y \leq 6$ (this is because $z \geq 0$ forces $x + y \leq 6$). R is a triangle in the first quadrant formed by the three segments segments

$$y = 0; \quad 0 \leq x \leq 6$$
$$x = 0; \quad 0 \leq y \leq 6$$
$$y = -x + 6; \quad 0 \leq x \leq 6$$

Step 1: Find critical points in the interior of R. Note $P = 6xy - x^2y - xy^2$, so $P_x = 6y - 2xy - y^2$ and $P_y = 6x - x^2 - 2xy$. Ignoring the points with $x = 0$ or $y = 0$, which are not in the interior of R, solving the system $6y - 2xy - y^2 = 0$, $6x - x^2 - 2xy = 0$ is the same as solving the system $6 - 2x - y = 0$, $6 - x - 2y = 0$. Solving the first equation for y and then substituting into the second equation gives $x = 2$. Substituting this back into either equation gives $y = 2$, so our critical point is $(2, 2)$. At this point, $P = (2)(2)(6 - 2 - 2) = 8$.

Step 2: Find the maximums and minimums on the boundary of R. On the segment $y = 0; \quad 0 \leq x \leq 6$, $P = x(0)(6 - x - y) = 0$, so this segment has an absolute max and minimum value of 0. Similarly, on the segment $x = 0; \quad 0 \leq y \leq 6$, $P = (0)y(6 - x - y) = 0$, so this segment has an absolute max and minimum value of 0. Finally, on the segment $y = -x + 6; \quad 0 \leq x \leq 6$, $P = xy(6 - x - (-x + 6)) = 0$, so this segment also has an absolute max and minimum value of 0.

Putting it all together, we find that P has an absolute maximum value on R of 8.

8. (10 pts) A rectangular box has dimensions $x = 3$ meters, $y = 2$ meters, $z = 1$ meter. If x and y are increasing at 1 cm/min and 2 cm/min, respectively, while z is decreasing at 2 cm/min, is the volume of the block increasing or decreasing at that instant? At what rate, is the volume changing?

Let V be the volume of the box. Then $V = xyz$, and we want to find $\frac{dV}{dt}$. By the chain rule,

$$\begin{aligned} \frac{dV}{dt} &= \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dv}{dt} \\ &= yz\frac{dx}{dt} + xz\frac{dy}{dt} + xy\frac{dz}{dt} \end{aligned}$$

We know $x = 3$ meters, $y = 2$ meters, $z = 1$ meter and $\frac{dx}{dt} = 1$ cm/min, $\frac{dy}{dt} = 2$ cm/min, $\frac{dz}{dt} = -2$ cm/min. Before plugging in, we must choose a common unit of length. Choosing cm, we get $x = 300$ cm, $y = 200$ cm, $z = 100$ cm. Now we plug in to the equation for $\frac{dV}{dt}$ to get $\frac{dV}{dt} = -40000 \text{ cm}^3/\text{min}$, so the volume is decreasing (because the rate is negative) at a rate of $40000 \text{ cm}^3/\text{min}$. Alternatively, the volume is decreasing at a rate of $0.04 \text{ meters}^3/\text{min}$.