MATH 4510: Introduction to Probability

October 6, 2008

Midterm

I have neither given nor received aid on this exam.

SOLUTIONS

In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	10 pts	
2	10 pts	
3	15 pts	
4	15 pts	
Total	50 pts	

1. (10 pts) Seven cards are randomly chosen, without replacement, from a standard 52 card deck. Find the probability that at least one of each suit is chosen. (Hint: Let S, H, D and C denote, respectively, the events that there are no spades, no hearts, no diamonds and no clubs chosen. What does $P(S \cup H \cup D \cup C)$ represent? Compute it.)

SOLUTION:

Since $P(S \cup H \cup D \cup C)$ represents the probability that at least one of the suits does *not* appear in the seven cards. Thus, $1 - P(S \cup H \cup D \cup C)$ gives the probability that we are interested in.

Now,

$$\begin{aligned} P(S \cup H \cup D \cup C) &= P(S) + P(H) + P(D) + P(C) \\ &\quad - P(SH) - P(SD) - P(SC) - P(HD) - P(HC) - P(DC) \\ &\quad + P(SHD) + P(SHC) + P(SDC) + P(HDC) \\ &\quad - P(SHDC), \end{aligned}$$

and it is easily seen that

$$P(S) = P(H) = P(D) = P(C) = \frac{\binom{39}{7}}{\binom{52}{7}},$$

$$P(SH) = P(SD) = P(SC) = P(HD) = P(HC) = P(DC) = \frac{\binom{26}{7}}{\binom{52}{7}},$$

$$P(SHD) = P(SHC) = P(SDC) = P(HDC) = \frac{\binom{13}{7}}{\binom{52}{7}},$$

and $P(SHDC) = 0$.

Thus,

$$P(S \cup H \cup D \cup C) = \frac{4\binom{39}{7} - 6\binom{26}{7} + 4\binom{13}{7}}{\binom{52}{7}}$$

and

$$\textbf{Prob}\{\text{at least one of each suit}\} = 1 - \left(\frac{4\binom{39}{7} - 6\binom{26}{7} + 4\binom{13}{7}}{\binom{52}{7}} \right).$$

2. (10 pts) Urn A has 5 white and 7 black balls. Urn B has 3 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn A is selected, whereas if the outcome is tails, a ball from urn B is selected. Suppose that a black ball is selected. What it the probability that the coin landed tails?

SOLUTION:

If we denote the possible outcomes of the coin by H and T, and the event of choosing from a colored ball by W and B, then we are interested in $P(T|B)$. By Bayes formula we have

$$P(T|B) = \frac{P(B|T)P(T)}{P(B|T)P(T) + P(B|H)P(H)}$$

Now, $P(T) = P(H) = \frac{1}{2}$ and $P(B|T) = \frac{12}{15}$ and $P(B|H) = \frac{7}{12}$. Thus,

$$P(T|B) = \frac{12/15}{12/15 + 7/12} = \frac{48}{83}$$

3. (15 pts) Suppose E and F are two events in a sample space S. Prove the following, making sure to explain each step.

(a) $P(E \cup F) = P(E) + P(F) - P(EF)$.

PROOF:

There are many ways to do this. One way is to notice that

$$E \cup F = EF^C \cup FE^C \cup EF,$$

and that the sets in the union on the right hand side are mutually exclusive. It follows that

$$\begin{aligned} P(E \cup F) &= P(EF^C) + P(FE^C) + P(EF) \\ &= P(EF^C) + P(EF) + P(FE^C) + P(EF) - P(EF). \end{aligned}$$

Now since EF^C and EF are mutually exclusive (as are FE^C and EF) we have

$$P(E \cup F) = P(EF^C \cup EF) + P(FE^C \cup EF) - P(EF).$$

Finally, we notice that $EF^C \cup EF = E$ and $FE^C \cup EF = F$, which proves the identity.

(b) $P(E) = P(E|F)P(F) + P(E|F^C)\left[1 - P(F)\right]$.

PROOF: We start by writing

$$P(E) = P(EF) + P(EF^C),$$

then by the definition of conditional probability we have $P(EF) = P(E|F)P(F)$ and $P(EF^C) = P(E|F^C)P(F^C)$. Thus,

$$P(E) = P(E|F)P(F) + P(E|F^C)P(F^C).$$

The assertion follows since $P(F^C) = 1 - P(F)$.

4. (15 pts) An urn has r red and w white balls that are randomly removed one at a time. Let R_i be the event that the ith ball removed is red. Find

(a) $P(R_i)$

SOLUTION: This is 5 on the Chapter 3 Self-Test.

Since each of the $r + w$ balls is equally likely to be the ith ball removed,

$$P(R_i) = \frac{r}{r+w}$$

(b) $P(R_5|R_3)$

SOLUTION:

$$P(R_j|R_i) = \frac{P(R_iR_j)}{P(R_i)} = \frac{\binom{r}{2}/\binom{r+w}{2}}{r/(r+w)} = \frac{r-1}{r+w-1}$$

(c) $P(R_3|R_5)$

SOLUTION: The answer is the same for part b:

$$\frac{r-1}{r+w-1}$$