MATH 2400: CALCULUS 3

February 14, 2007

MIDTERM 1

I have neither given nor received aid on this exam.

Name:____

○ **001** E. Kim .... (9am)

○ **002** E. Angel ....(10am)

○ **003** I. Mishev .... (11am)

○ **004** J. Boisvert .... (12am)

○ **005** A. Gorokhovsky ...... (1pm)

If you have a question raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give adequate explanations.

DO NOT WRITE IN THIS BOX!

Problem	**Points**	**Score**
1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
6	20 pts	
7	20 pts	
8	20 pts	
TOTAL	160 pts	

1. (9 pt.) Let $\mathbf{v} = \langle 1, 1, 1\rangle$, $\mathbf{b} = \langle 1, 0, 1\rangle$. Find the orthogonal projection of $\mathbf{v}$ onto $\mathbf{b}$.

$$\text{proj}_{\mathbf{b}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{b}}{||\mathbf{b}||^2}\mathbf{b} = \frac{\langle 1,1,1\rangle \cdot \langle 1,0,1\rangle}{||\langle 1,0,1\rangle||^2}\langle 1,0,1\rangle = \frac{2}{\left(\sqrt{2}\right)^2}\langle 1,0,1\rangle = \langle 1,0,1\rangle$$

2. (9 pt.) Find an equation of the plane passing through $(-1, 0, 1)$ and perpendicular to the line $\langle x, y, z\rangle = \langle 5t, 1 + t, -t\rangle$.

Let $\mathbf{n} = \langle 5, 1, -1\rangle$ (a direction vector for the line). Then $\mathbf{n}$ will serve as a normal vector for our plane, since our plane is to be perpendicular to the line. This gives an equation for the plane to be $5(x + 1) + 1(y - 0) - 1(z - 1) = 0$, which simplifies to $5x + y - z + 6 = 0$.

3. (9 pt.) Determine whether the lines L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

$$\begin{aligned} L_1 &: x = -6t, \quad y = 1 + 9t, \quad z = -3t \\ L_2 &: x = 1 + 2t, \quad y = 4 - 3t, \quad z = t \end{aligned}$$

L_1 has $\mathbf{b}_1 = \langle -6, 9, -3\rangle$ as a direction vector, and L_2 has $\mathbf{b}_2 = \langle 2, -3, 1\rangle$ as a direction vector. Since $\mathbf{b}_1$ is a scalar multiple of $\mathbf{b}_2$ ($\mathbf{b}_1 = -3\mathbf{b}_2$), the lines are parallel. **However**, the lines may also intersect, but this only happens if the lines are the same. Since $(0, 1, 0)$ is on L_1 and not on L_2, the lines are not the same. We conclude that the lines are parallel and non-intersecting.

4.

(a) (9 pt.) Find parametric equations of the tangent line to the parametric curve $\mathbf{r}(t) = \langle 2+t^3, 1-4t, 5-t^2\rangle$ at the time $t = 1$.

$\mathbf{r}'(t) = \langle 3t^2, -4, -2t\rangle$, so a direction vector for the tangent line at $t = 1$ is given by $\mathbf{r}'(1) = \langle 3, -4, -2\rangle$. Also, $\mathbf{r}(1) = \langle 3, -3, 4\rangle$ 'represents' a point on the line. Thus, an equation for the tangent line (in vector form) at $t = 0$ is given by $\langle 3, -3, 4\rangle + t\langle 3, -4, -2\rangle$. In parametric form, we get, $x = 3+3t$, $y = -3-4t$, $z = 4-2t$.

(b) (9 pt.) Find the point of intersection of this line with the xy-plane.

A curve intersects the xy-plane, when $z = 0$. Solving $0 = 4-2t$, we see that we intersect the xy-plane when $t = 2$, so the point of intersection (found by plugging $t = 2$ into the parametric equations for the line) is $(9, -11, 0)$.

5. (10 pt.) Find the equation of the sphere centered at $(0, 1, 5)$ and tangent to the plane $3x + 6y - 2z - 5 = 0$.

Since we have the center of the sphere, we need only find its radius. Since the sphere is tangent to the plane $3x + 6y - 2z - 5 = 0$, the radius will be the shortest distance from the center of the sphere to the plane. However, the **shortest distance** from a point, $P(x_0, y_0, z_0)$, to a plane, $ax + by + cz + d = 0$, is exactly how we defined the **distance** from the point to the plane, which is given by the formula $\textit{distance} = \dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$. This tells us that the radius of the sphere is $r = \dfrac{|3(0) + 6(1) - 2(5) - 5|}{\sqrt{3^2+6^2+2^2}} = \dfrac{9}{7}$, so the equation of the sphere is $(x-0)^2 + (y-1)^2 + (z-5)^2 = \left(\frac{9}{7}\right)^2$. This simplifies to $x^2 + (y-1)^2 + (z-5)^2 = \frac{81}{49}$.

6. (9 pt.) Find the area of the triangle defined by the points $P_1(-1,0,0)$, $P_2(0,1,0)$, $P_3(1,1,1)$.

Consider the vectors $\overrightarrow{P_1P_2} = \langle 1,1,0\rangle$ and $\overrightarrow{P_1P_3} = \langle 2,1,1\rangle$. The area of the triangle formed by the points will be $\frac{1}{2}$ the area of the parallelogram with $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_2}$ as adjacent sides. We know that the area of the parallelogram is $||\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2}||$, so the area of the triangle is $\frac{1}{2}||\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_2}|| = \frac{1}{2}||\langle 1,-1,-1\rangle|| = \frac{1}{2}\sqrt{3} = \frac{\sqrt{3}}{2}$.

7. (18 pt.) Match the equations. Match each rectangular equation from the first column with an equivalent cylindrical equation in the second column. Then match each cylindrical equation in the second column with an equivalent spherical equation in the third column.

(1) $x^2+y^2+z^2=9$
(2) $z=3x^2+3y^2$
(3) $(x-1)^2+y^2+z^2=1$

(a) $z=3r^2$
(b) $r^2+z^2=9$
(c) $r^2+z^2-2r\cos\theta=0$

(i) $\rho-2\sin\phi\cos\theta=0$
(ii) $\rho=3$
(iii) $\cos\phi-3\rho\sin^2\phi=0$

(1)-(b)-(ii). Since $x^2+y^2=r^2$ (1) matches with (b). Since $r^2+z^2=\rho^2$, (b) matches with $\rho^2=9$ which matches with (ii).

(2)-(a)-(iii). Since $x^2+y^2=r^2$ (1) matches with (a). Since $z=\rho\cos\phi$ and $r=\rho\sin\phi$, (a) matches with $\rho\cos\phi=3(\rho\sin\phi)^2$ which matches with (iii).

(3)-(c)-(i). Note that $(x-1)^2+y^2+z^2=1$ expands as $x^2-2x+1+y^2+z^2=1$. Since $x^2+y^2=r^2$ and $x=r\cos\theta$, (3) matches with (c). Using $z=\rho\cos\phi$ and $r=\rho\sin\phi$, we can make (c) match with (i).

8. (9 pt.) Solve the initial value problem $\mathbf{r}'(t) = \mathbf{i} + e^t\,\mathbf{j} + \frac{1}{t+1}\,\mathbf{k}$, $\mathbf{r}(0) = \mathbf{0}$.

Integrating term by term we get $\mathbf{r}(t) = \langle t, e^t, \ln|t+1|\rangle + \mathbf{C}$. Since $\mathbf{r}(0) = \mathbf{0}$, we find $\langle 0, e^0, \ln|1|\rangle + \mathbf{C} = \mathbf{0}$. So $\mathbf{C} = \langle 0, -1, 0\rangle$. Thus, $\mathbf{r}(t) = \langle t, e^t, \ln|t+1|\rangle + \langle 0, -1, 0\rangle = \langle t, e^t - 1, \ln|t+1|\rangle$

9. (9 pt.) Find the arc length parametrization of the parametric curve $x = \frac{1}{2}t$, $y = \frac{1}{3}(1-t)^{\frac{3}{2}}$, $z = \frac{1}{3}(1+t)^{\frac{3}{2}}$, $-1 \le t \le 1$ that has the same orientation as the given curve. Use the point when $t = 0$ as the reference point.

We first calculate

$$\begin{aligned}
s &= \int_0^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du \\
&= \int_0^t \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}(1-u)^{\frac{1}{2}}\right)^2 + \left(\frac{1}{2}(1+u)^{\frac{1}{2}}\right)^2}\, du \\
&= \int_0^t \sqrt{\frac{1}{4} + \frac{1}{4}(1-u) + \frac{1}{4}(1+u)}\, du \\
&= \int_0^t \frac{\sqrt{3}}{2}\, du \\
&= \frac{\sqrt{3}}{2} t
\end{aligned}$$

So $t = \frac{2}{\sqrt{3}}s$. Substituting back into our parametric equations, we get $x = \frac{1}{\sqrt{3}}s$, $y = \frac{1}{3}(1 - \frac{2}{\sqrt{3}}s)^{\frac{3}{2}}$, $z = \frac{1}{3}(1 + \frac{2}{\sqrt{3}}s)^{\frac{3}{2}}$, $-\frac{\sqrt{3}}{2} \le t \le \frac{\sqrt{3}}{2}$