Midterm #2

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your name in the upper right hand corner of each page.

PROBLEMS:

1. Let M and N be closed linear subspaces of a Hilbert space $\mathcal{H}$, and assume P, Q are orthogonal projections with $\operatorname{ran}(P) = M$ and $\operatorname{ran}(Q) = N$. Show that the following conditions are equivalent:
   (a) $M \subset N$
   (b) $QP = P$
   (c) $PQ = P$
   (d) $\|Px\| \leq \|Qx\|$ for all $x \in \mathcal{H}$
   (e) $(x, Px) \leq (x, Qx)$ for all $x \in \mathcal{H}$
2. Consider the Schrödinger equation on the circle,
$$\begin{aligned} iu_t &= u_{xx}, \qquad x \in \mathbb{T}, \quad t \in \mathbb{R} \\ u(x,0) &= f(x), \qquad x \in \mathbb{T}, \end{aligned}$$
   where $u : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $f : \mathbb{T} \to \mathbb{C}$, and the derivatives are interpreted in an appropriate sense.
   (a) Solve for $u(x,t)$ by the use of Fourier series.
   (b) Briefly compare the smoothing property of the Schrödinger equation with that of the heat equation.
3. Suppose that $g : [0,1] \to \mathbb{C}$ is a continuous function. Define the multiplication operator
$$M : L^2([0,1], m) \to L^2([0,1], m)$$
   by
$$(Mg)(x) = g(x)f(x).$$
   (a) Prove that M is a bounded linear operator on $L^2([0,1], m)$ and compute the adjoint M^*.
   (b) Describe the point spectrum of M.
   (c) For what functions g is M self-adjoint? For what functions g is M unitary?
4. Suppose that $\{P_n\}$ is a sequence of orthogonal projections on a Hilbert space $\mathcal{H}$ such that
$$\operatorname{ran}(P_n) \subset \operatorname{ran}(P_{n+1}), \qquad \bigcup_{n=1}^{\infty} \operatorname{ran}(P_n) = \mathcal{H}.$$
   (a) Prove that for every $x \in \mathcal{H}$, $P_n x \to x$ as $n \to \infty$.

(b) Show that $\{P_n\}$ does not converge to the identity operator I with respect to the operator norm unless $P_n = I$ for all sufficiently large n.

5. Suppose that U is a unitary operator on a Hilbert space $\mathcal{H}$. Let $M = \{x \in \mathcal{H} : Ux = x\}$, and P the orthogonal projection onto M. For all $x \in \mathcal{H}$, show that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} U^n x = Px,$$

where U^0 represents the identity operator. This is known as **von Neumann's ergodic theorem**.