Midterm #1

INSTRUCTIONS:

1. Answer each question on a separate page. Turn in a page for each problem even if you cannot do the problem.
2. Label each answer sheet with the problem number.
3. Put your name in the upper right hand corner of each page.

PROBLEMS:

1. Our definition of Lebesgue points given in class applies to individual integrable functions, not to equivalence classes as we usually define $L^1(\mathbb{R}^n)$. If $F \in L^1(\mathbb{R}^n)$ is one of these equivelence classes, one may call a point $x \in \mathbb{R}^n$ a Lebesgue point of F if there is a complex number, let us call it $(SF)(x)$, such that

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - (SF)(x)| dm(y) = 0$$

   for one (and hence for every) $f \in F$. Define $SF(x)$ to be 0 at those points $x \in \mathbb{R}^n$ which are not Lebesgue points of F. If $f \in F$, and x is a Lebesgue point of f, show that x is also a Lebesgue point of F, and $f(x) = (SF)(x)$. Hence, $SF \in F$. (This can be viewed as showing that S "selects" a member of F that has maximal set of Lebesgue points.)

2. Suppose ν is a signed measure and $\nu = \lambda_1 - \lambda_2$ for some positive measures λ_1, λ_2. Prove that $\lambda_1 \geq \nu^+$ and $\lambda_2 \geq \nu^-$. This is referred to as the **minimum property of the Jordan decomposition**.
3. Show that the product of two absolutely continuous functions on $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.
4. A subset A of a Hilbert space $\mathcal{H}$ is called **convex** if for all $x, y \in A$ and every $t \in (0, 1)$, $tx + (1-t)y \in A$. Prove that every nonempty closed convex subset of a Hilbert space has a unique element of smallest norm.
5. If A is a subset of a Hilbert space, prove that

$$A^\perp = \overline{A}^\perp,$$

   where $\overline{A}$ is the closure of A. If $\mathcal{M}$ is a linear subspace of a Hilbert space, show that

$$(\mathcal{M}^\perp)^\perp = \overline{\mathcal{M}}.$$