Homework #6

1. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $T \in \mathcal{S}^*(\mathbb{R}^n)$, prove that $\varphi * T \in C^\infty(\mathbb{R}^n)$.
2. If $\varphi \in \mathcal{S}(\mathbb{R})$, prove that
$$\varphi\delta' = \varphi(0)\delta' - \varphi'(0)\delta,$$
where $\delta \in \mathcal{S}^*(\mathbb{R})$ is the delta function.
3. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the function
$$f_n(x) = \begin{cases} n^2 & \text{if } -1/n < x < 0, \\ -n^2 & \text{if } 0 < x < 1/n, \\ 0 & \text{otherwise.} \end{cases}$$
Show that the sequence of regular distributions $\{T_{f_n}\}$ converges in $\mathcal{S}^*(\mathbb{R})$ as $n \to \infty$, and determine its distributional limit.
4. (a) Give an example of a function $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is not continuous. Why does this not contradict the Riemann–Lebesgue lemma?
   (b) Give an example of a function $f \in L^1(\mathbb{R})$ such that $\hat{f} \notin L^1(\mathbb{R})$.
5. Let $\varphi : \mathbb{R} \to \mathbb{C}$ be any Schwartz function with the property that
$$\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = 1,$$
and define
$$a = \int_{-\infty}^{\infty} x^2 |\varphi(x)|^2 dx, \qquad b = \int_{-\infty}^{\infty} \xi^2 |\hat{\varphi}(\xi)|^2 d\xi.$$
Prove the **Heisenberg uncertainty principle**:
$$ab \geq \frac{1}{4}.$$